Quasi-elliptic cohomology

Zhen Huan

University of Illinois at Urbana-Champaign

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Overview

Plan.

- Motivation and construction
- The power operation
- The orthogonal $G-$spectra

Motivation

An old idea of Witten [Landweber]

The elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of $LX = \mathbb{C}^{\infty}(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops.

It's surprisingly difficult to make this precise.

Why?

In application, one needs to consider the case that a group G acts on X. In this case the loop space LX has rich structures as an orbifold.

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Bibundles $\sim$ "bimodules" in geometry

Bibundles combine several widely used notions, including smooth maps, Lie homomorphisms, and principal bundles.

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ [Schommer-Pries] [Lerman]

a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0 \times H_0}$
  - $\mathbb{G}_1 \, {}_s\!\times_\tau P \longrightarrow P$;
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such that

- $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$; • $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$; • $g \cdot (p \cdot h) = (g \cdot p) \cdot h$
- $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
- $\mathbb{G}_1 \, {}_s\!\times_\tau P \longrightarrow P \, {}_\sigma\!\times_\sigma P$ $\quad (g, p) \mapsto (g \cdot p, p)$ is an isomorphism.

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- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

- Morphisms:

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ & \nwarrow^{\pi'} & \downarrow{\scriptstyle\alpha} & \nearrow_{f'} & \\ & & P' & & \end{array}$$

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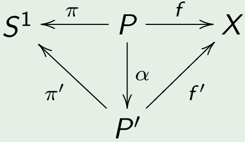

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Quasi-elliptic cohomology

The isotropy groups in $Loop^{ext}(X /\!\!/ G)$ may be infinite dimensional topological groups when G is not finite.

the subgroupoid $\Lambda(X /\!\!/ G)$ instead

$$\Lambda(X /\!\!/ G) := \coprod_{g \in G^{tors}_{conj}} X^g /\!\!/ \Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors};

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

$QEll$ as equivariant $K-$theories

$$QEll_G(X) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

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Power Operation

Quasi-elliptic cohomology has power operations, which gives it the structure of an "$H_\infty-$ring theory" [Ganter 06].

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X/\!/G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^\pm]} n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)} :$$

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$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^\Lambda_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

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Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.

When $n = 2$,

$$QEll_{\Sigma_2}(X \times X) \cong K(X \times X)[q^{\pm}][1, s]/(s^2 - 1) \times K(X)[q^{\pm}][y]/(y^2 - q)$$

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

A Ring Homomorphism

$\overline{P}_N : QEll_G(X) \xrightarrow{\mathbb{P}_N} QEll_{G \wr \Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G \times \Sigma_N}(X^{\times N}) \xrightarrow{diag^*}$
$QEll_{G \times \Sigma_N}(X) \cong QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt}) \longrightarrow$
$QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr}$

- analogous to the Adams operations of equivariant K-theories.
- but different and new.

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$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) =$
$(x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

A Ring Homomorphism

$\overline{P}_N : QEll_G(X) \xrightarrow{\mathbb{P}_N} QEll_{G\wr\Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G\times\Sigma_N}(X^{\times N}) \xrightarrow{diag^*}$
$QEll_{G\times\Sigma_N}(X) \cong QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt}) \longrightarrow$
$QEll_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr}$

- analogous to the Adams operations of equivariant K-theories.
- but different and new.

Finite Subgroups of Tate Curve

Theorem (Huan)

$$QEll(pt/\!\!/\Sigma_N)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e\rangle,$$

where $\mathcal{I}_{tr}^{QEll}$ is the transfer ideal and q' is the image of q under the power operation $\mathbb{P}_N$. The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Theorem (Huan)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q'_s is the image of q under the stringy power operation P_N^{string}, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

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Goerss-Hopkins-Miller theorem constructs many example of $E_\infty-$rings which represent elliptic cohomology theories, including Tate K-theory.

Question

Can we construct $E_\infty - G-$spectrum which represents equivariant elliptic cohomology theory (e.g. $G-$equivariant Tate K-theory)?

Orthogonal $G-$spectra of quasi-elliptic cohomology [Huan]

We construct a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$. For each faithful $G-$representation V, $E(G,V)$ weakly represents $QEll_G^V(-)$ in the sense

$$\pi_k(E(G,V)) = QEll_G^V(S^k), \text{ for each } k.$$

Can $E(G,-)$ arise from an orthogonal spectrum?

No.

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not trivial.

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Global Homotopy Theory [Schwede][May]

Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

$\Rightarrow$ **global homotopy theory**

Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

Almost Global Homotopy Theory [Huan]

- an extension of global homotopy theory;
- classifies those theories that are almost "global";
- the restriction maps are equivariant weak equivalence.

We can define global quasi-elliptic cohomology. [Huan]

Combining the orthogonal $G-$spectra $\{E(G,-)\}$, we get an ultra-commutative global ring spectrum in the new theory.

a new Global Homotopy Theory

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Model Structure on the almost global spaces

We formulate several model structures and are formulating the one below.

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Thank you.

Some references

http://www.math.uiuc.edu/~huan2/Zhen-AMS-2017-Slides.pdf

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