Quasi-elliptic cohomology

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Overview

Plan.

- Motivation and construction
- The power operation
- The orthogonal $G-$spectrum

Motivation

An old idea of Witten [Landweber]

The elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops.

It's surprisingly difficult to make this precise.

Why?

In application, one needs to consider the case that a group G acts on X. In this case the loop space LX has rich structures as an orbifold.

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The category of bibundles

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ [Schommer-Pries][Lerman]

a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0 \times H_0}$
  - $\mathbb{G}_1 \;{}_s\!\times_\tau P \longrightarrow P$;
  - $P \;{}_\sigma\!\times_t \mathbb{H}_1 \longrightarrow P$

such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 \;{}_s\!\times_t \mathbb{G}_1 \;{}_s\!\times_\tau P$;
2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P \;{}_\sigma\!\times_t \mathbb{H}_1 \;{}_s\!\times_t \mathbb{H}_1$;
3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 \;{}_s\!\times_\tau P \;{}_\sigma\!\times_t \mathbb{H}_1$.
5. $\mathbb{G}_1 \;{}_s\!\times_\tau P \longrightarrow P \;{}_\sigma\!\times_\sigma P$ $\quad (g, p) \mapsto (g \cdot p, p)$ is an isomorphism.

Bibundle Map [Schommer-Pries][Lerman]

a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ commuting with the $\mathbb{G}-$ and $\mathbb{H}-$action.

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The Loop Spaces of Interest

Example ($Loop_1(X/\!\!/G) := Bibun(S^1/\!\!/*, X/\!\!/G)$)

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

Example (Another Model $Loop_2(X/\!\!/G)$)

- Objects: (σ, γ)
  - $\sigma \in G$
  - $\gamma : \mathbb{R} \longrightarrow X$ smooth $\gamma(s+1) = \gamma(s) \cdot \sigma$
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$\mathbb{T}-$action

Example ($Loop_1^{ext}(X/\!\!/G)$)

- the same objects as $Loop_1(X/\!\!/G)$;
- $(t,\alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
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- the same objects as $Loop_2(X/\!\!/G)$;
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$$(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

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Good Groupoid or Not?

A skeleton of $Loop_2^{ext}(X /\!\!/ G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X /\!\!/ G)$.
- $L_g G = \{\alpha : \mathbb{R} \longrightarrow G | \alpha(s+1) = g^{-1}\alpha(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\alpha, t) \cdot (\alpha', t') := (s \mapsto \alpha(s)\alpha'(s+t), t+t')$.
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$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X /\!\!/ L_g G \rtimes \mathbb{T}$ is a skeleton of $Loop_2^{ext}(X /\!\!/ G)$.

Problem

$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite.

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Consider the subgroupoid $\Lambda(X /\!\!/ G)$ instead

$$\Lambda(X /\!\!/ G) := \coprod_{g \in G^{tors}_{conj}} X^g /\!\!/ \Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors};

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

$\Lambda_G(g)$ acts on X^g by

$$[h, t] \cdot x := h \cdot x.$$

$QEll$ as equivariant $K-$theories

$$QEll_G(X) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

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Restriction

$\phi : X/\!\!/G \longrightarrow Y/\!\!/H \Longrightarrow \Lambda(\phi) : \Lambda(X/\!\!/G) \longrightarrow \Lambda(Y/\!\!/H)$

$$\begin{array}{ccc} QEll^*(Y/\!\!/H) & \xrightarrow{\phi^*} & QEll^*(X/\!\!/G) \\ \downarrow{\scriptstyle \pi_{\phi(\tau)}} & & \downarrow{\scriptstyle \pi_\tau} \\ K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^\tau) \end{array}$$

Künneth Map

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Setting up the theory

Restriction

$$\phi : X /\!/ G \longrightarrow Y /\!/ H \Longrightarrow \Lambda(\phi) : \Lambda(X /\!/ G) \longrightarrow \Lambda(Y /\!/ H)$$

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Setting up the theory

The change-of-group isomorphism

- H: closed subgroup of G;
- X: H-space;
- $\phi : H \longrightarrow G$ is the inclusion.

Theorem

The change-of-group map ρ_H^G is an isomorphism.

$$\rho_H^G : QEll_G^*(G \times_H X) \xrightarrow{\phi^*} QEll_H^*(G \times_H X) \xrightarrow{i^*} QEll_H^*(X)$$

- ϕ^*: *the restriction map*
- $i : X \longrightarrow G \times_H X$: $i(x) = [e, x]$.

Induced map

$$\mathcal{I}_H^G : QEll(X /\!\!/ H) \xrightarrow{\cong} QEll((G \times_H X) /\!\!/ G) \longrightarrow QEll(X /\!\!/ G)$$

- the first map: the change-of-group isomorphism
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Power Operation

Quasi-elliptic cohomology has power operations, which gives it the structure of an "H_∞−ring theory" [Ganter 06].

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X/\!\!/G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} n}$ defines an operation

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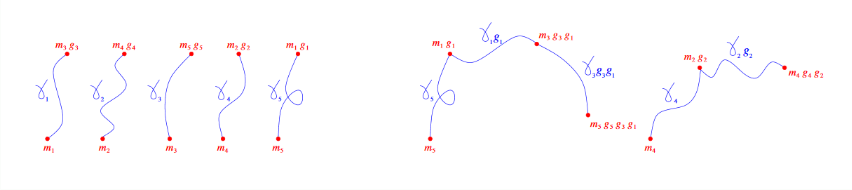
$\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ and $\prod\limits_{k}\prod\limits_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}}X$ are $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)$−equivariant homeomorphic.

Example ($\mathcal{L}_{(g_1,\cdots g_5,(135)(24))}(X^{\times 5})$ and ${}_3\mathcal{L}_{g_5g_3g_1}(X)\times {}_2\mathcal{L}_{g_4g_2}(X)$)



This picture is from [Ganter].

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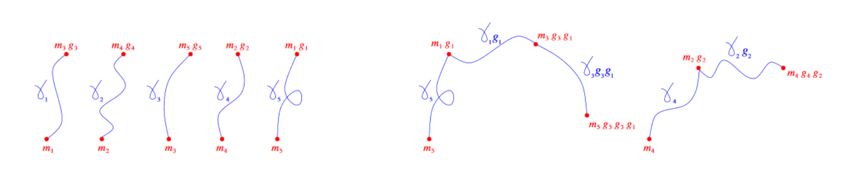

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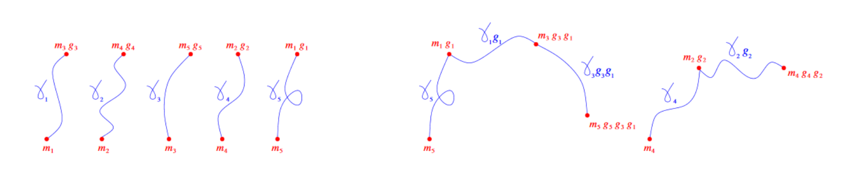

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Note

$$\Lambda^1_G(g, g) = \Lambda_G(g).$$

A groupoid equivalent to $\Lambda(X /\!\!/ G)$

- objects $\coprod_{g \in G^{tors}} X^g$;
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$\Lambda^{var}_{(\underline{g},\sigma)}(X)$ and the Functor $(\ )^{\Lambda}_{k}$: Dilating the Loops

Let $\Lambda^{var}_{(\underline{g},\sigma)}(X)$ be the groupoid with

- the same objects as $\Lambda_{(\underline{g},\sigma)}(X)$
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- identity on objects
- sends each $[g,t] \in \Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$ to $[g,\frac{t}{k}] \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$.

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Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.

When $n = 2$,

$$QEll_{\Sigma_2}(X \times X) \cong K(X \times X)[q^{\pm}][1, s]/(s^2 - 1) \times K(X)[q^{\pm}][y]/(y^2 - q)$$

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

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Can the classification of the finite subgroups of an elliptic curve be given by the associated elliptic cohomology theory?

Morava E-theory [Strickland]

The Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of the formal group $\mathbb{G}_E$.

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$$E_q : y^2 + xy = x^3 + a_4 x + a_6$$

$$a_4 = -5\sum_{n\geqslant 1} n^3 q^n/(1-q^n) \qquad a_6 = -\frac{1}{12}\sum_{n\geqslant 1}(7n^5+5n^3)q^n/(1-q^n).$$

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The finite subgroups of the Tate curve are the kernels of isogenies. Its finite subgroups of order N can be classified by

$$\mathcal{O}_{Sub_N} = \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle.$$

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Orthogonal $G-$spectrum [Mandell and May]

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

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A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

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An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

- a unit $\eta : S \longrightarrow X$;
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Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

$\Rightarrow$ **global homotopy theory**

Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

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Orthogonal Space [Schwede]

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Goerss-Hopkins-Miller theorem constructs many examples of $E_\infty-$rings which represent elliptic cohomology theories, including Tate K-theory.

Question

Can we construct $E_\infty - G-$spectrum which represents equivariant elliptic cohomology theory (e.g. $G-$equivariant Tate K-theory)?

Orthogonal $G-$spectrum of quasi-elliptic cohomology [Huan]

We construct a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$. For each faithful $G-$representation V, $E(G,V)$ weakly represents $QEll_G^V(-)$ in the sense

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- $S(G,V)_g := Sym(V) \setminus Sym(V)^g$.
- KU: the global complex $K-$spectrum;
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$F_g(G,V) := \mathrm{Map}_{\mathbb{R}}(S^{(V)_g}, KU((V)_g \oplus V^g))$
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- an extension of global homotopy theory;
- classifies those theories that are almost "global";
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We can define global quasi-elliptic cohomology. [Huan]

Combining the orthogonal $G-$spectra $\{E(G,-)\}$, we get an ultra-commutative global ring spectrum in the new theory.

We formulate several model structures and are formulating the one below.

Conjecture

There is a global model structure on the almost global spaces that is Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

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Thank you.

Some references

http://gagp.sysu.edu.cn/zhenhuan/Zhen-SUSTech-2017-Slides.pdf

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