Quasi-elliptic cohomology

Zhen Huan

University of Illinois at Urbana-Champaign

October 30, 2016

Overview

Plan.

- The construction of quasi-elliptic cohomology
- The power operation
- The spectra

Groupoids

Definition

A groupoid $\mathbb{G}$ is a small category in which each arrow is an isomorphism. That is, $\mathbb{G}$ consists of a set $\mathbb{G}_0$ of objects and a set $\mathbb{G}_1$ of arrows.

Example (Translation Groupoids)

Compact Lie $G \curvearrowright X$.
$X//G$: the groupoid with

- objects: points $x \in X$
- arrows: $\alpha : x \longrightarrow y$ those $\alpha \in G$ for which $\alpha \cdot x = y$.

Bibundle

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ [Schommer-Pries]

a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0 \times H_0}$
  - $\mathbb{G}_1 \, {}_s\!\times_\tau P \longrightarrow P$;
  - $P \, {}_\sigma\!\times_t \mathbb{H}_1 \longrightarrow P$

such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 \, {}_s\!\times_t \mathbb{G}_1 \, {}_s\!\times_\tau P$;
2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P \, {}_\sigma\!\times_t \mathbb{H}_1 \, {}_s\!\times_t \mathbb{H}_1$;
3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 \, {}_s\!\times_\tau P \, {}_\sigma\!\times_t \mathbb{H}_1$.
5. $\mathbb{G}_1 \, {}_s\!\times_\tau P \longrightarrow P \, {}_\sigma\!\times_\sigma P$ $\quad (g, p) \mapsto (g \cdot p, p)$ is an isomorphism.

Bibundle Map [Schommer-Pries]

a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ commuting with the $\mathbb{G}-$ and $\mathbb{H}-$action.

Bibundle

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ [Schommer-Pries]

a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0 \times H_0}$
  - $\mathbb{G}_1 \,{}_s\!\times_\tau P \longrightarrow P$;
  - $P \,{}_\sigma\!\times_t \mathbb{H}_1 \longrightarrow P$

such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 \,{}_s\!\times_t \mathbb{G}_1 \,{}_s\!\times_\tau P$;
2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P \,{}_\sigma\!\times_t \mathbb{H}_1 \,{}_s\!\times_t \mathbb{H}_1$;
3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 \,{}_s\!\times_\tau P \,{}_\sigma\!\times_t \mathbb{H}_1$.
5. $\mathbb{G}_1 \,{}_s\!\times_\tau P \longrightarrow P \,{}_\sigma\!\times_\sigma P$ $\quad (g, p) \mapsto (g \cdot p, p)$ is an isomorphism.

Bibundle Map [Schommer-Pries]

a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ commuting with the $\mathbb{G}-$ and $\mathbb{H}-$action.

Bibundle

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ [Schommer-Pries]

a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0 \times H_0}$
  - $\mathbb{G}_1 {}_s\times_\tau P \longrightarrow P$;
  - $P {}_\sigma\times_t \mathbb{H}_1 \longrightarrow P$

such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 {}_s\times_t \mathbb{G}_1 {}_s\times_\tau P$;
2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P {}_\sigma\times_t \mathbb{H}_1 {}_s\times_t \mathbb{H}_1$;
3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 {}_s\times_\tau P {}_\sigma\times_t \mathbb{H}_1$.
5. $\mathbb{G}_1 {}_s\times_\tau P \longrightarrow P {}_\sigma\times_\sigma P$ $\quad (g, p) \mapsto (g \cdot p, p)$ is an isomorphism.

Bibundle Map [Schommer-Pries]

a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ commuting with the $\mathbb{G}-$ and $\mathbb{H}-$action.

Bibundle

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ [Schommer-Pries]

a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0 \times H_0}$
  - $\mathbb{G}_1 \, {}_s\!\times_\tau P \longrightarrow P$;
  - $P \, {}_\sigma\!\times_t \mathbb{H}_1 \longrightarrow P$

such that

1. $g_1 \cdot (g_2 \cdot p) = (g_1 g_2) \cdot p$ for all $(g_1, g_2, p) \in \mathbb{G}_1 \, {}_s\!\times_t \mathbb{G}_1 \, {}_s\!\times_\tau P$;
2. $(p \cdot h_1) \cdot h_2 = p \cdot (h_1 h_2)$ for all $(p, h_1, h_2) \in P \, {}_\sigma\!\times_t \mathbb{H}_1 \, {}_s\!\times_t \mathbb{H}_1$;
3. $p \cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p)) \cdot p = p$ for all $p \in P$.
4. $g \cdot (p \cdot h) = (g \cdot p) \cdot h$ for all $(g, p, h) \in \mathbb{G}_1 \, {}_s\!\times_\tau P \, {}_\sigma\!\times_t \mathbb{H}_1$.
5. $\mathbb{G}_1 \, {}_s\!\times_\tau P \longrightarrow P \, {}_\sigma\!\times_\sigma P$ $\quad (g, p) \mapsto (g \cdot p, p)$ is an isomorphism.

Bibundle Map [Schommer-Pries]

a map $P \longrightarrow P'$ over $\mathbb{H}_0 \times \mathbb{G}_0$ commuting with the $\mathbb{G}-$ and $\mathbb{H}-$action.

Example ($Loop_1(X//G) := Bibun(S^1//*, X//G)$)

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ & \nwarrow^{\pi'} & \downarrow{\alpha} & \nearrow_{f'} & \\ & & P' & & \end{array}$$

Example (Another Model $Loop_2(X//G)$)

- Objects: (σ, γ)
  - $\sigma \in G$
  - $\gamma : \mathbb{R} \longrightarrow X$ smooth $\gamma(s+1) = \gamma(s) \cdot \sigma$
- Morphism $(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$: $\alpha : \mathbb{R} \longrightarrow G$ smooth, $\gamma'(s) = \gamma(s)\alpha(s)$.

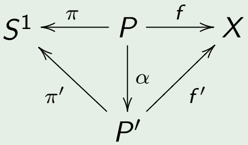
Example ($Loop_1(X//G) := Bibun(S^1//*, X//G)$)

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$



Example (Another Model $Loop_2(X//G)$)

- Objects: (σ, γ)
  - $\sigma \in G$
  - $\gamma : \mathbb{R} \longrightarrow X$ smooth $\gamma(s+1) = \gamma(s) \cdot \sigma$
- Morphism $(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$: $\alpha : \mathbb{R} \longrightarrow G$ smooth, $\gamma'(s) = \gamma(s)\alpha(s)$.

Examples of Bibundle

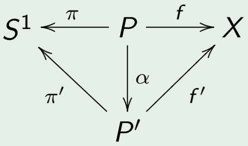
Example ($Loop_1(X//G) := Bibun(S^1//*, X//G)$)

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$



Example (Another Model $Loop_2(X//G)$)

- Objects: (σ, γ)
  - $\sigma \in G$
  - $\gamma : \mathbb{R} \longrightarrow X$ smooth $\gamma(s+1) = \gamma(s) \cdot \sigma$
- Morphism $(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$: $\alpha : \mathbb{R} \longrightarrow G$ smooth, $\gamma'(s) = \gamma(s)\alpha(s)$.

Examples of Bibundle

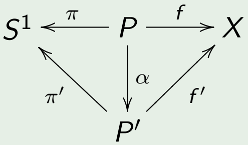
Example ($Loop_1(X//G) := Bibun(S^1//*, X//G)$)

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - • π : principal $G-$bundle over S^1
  - • f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$



Example (Another Model $Loop_2(X//G)$)

- Objects: (σ, γ)
  - • $\sigma \in G$
  - • $\gamma : \mathbb{R} \longrightarrow X$ smooth $\gamma(s+1) = \gamma(s) \cdot \sigma$
- Morphism $(\sigma, \gamma) \longrightarrow (\sigma', \gamma')$: $\alpha : \mathbb{R} \longrightarrow G$ smooth, $\gamma'(s) = \gamma(s)\alpha(s)$.

S^1−action

Example ($Loop_1^{ext}(X//G)$)

- the same objects as $Loop_1(X//G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: G−bundle map

Example ($Loop_2^{ext}(X//G)$)

- the same objects as $Loop_2(X//G)$;
- A morphism

$$(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

  - $\alpha : \mathbb{R} \longrightarrow G$ smooth
  - $t \in \mathbb{R}$

$$\gamma'(s) = \gamma(s - t)\alpha(s - t).$$

Example ($Loop_1^{ext}(X//G)$)

- the same objects as $Loop_1(X//G)$;
- $(t,\alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ \downarrow t & & \downarrow \alpha & \nearrow f' & \\ S^1 & \xleftarrow[\pi']{} & P' & & \end{array}$$

Example ($Loop_2^{ext}(X//G)$)

- the same objects as $Loop_2(X//G)$;
- A morphism

$$(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

- $\alpha : \mathbb{R} \longrightarrow G$ smooth
- $t \in \mathbb{R}$

$\gamma'(s) = \gamma(s-t)\alpha(s-t).$

Example ($Loop_1^{ext}(X//G)$)

- the same objects as $Loop_1(X//G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle \alpha} & \nearrow{\scriptstyle f'} & \\ S^1 & \xleftarrow[\pi']{} & P' & & \end{array}$$

Example ($Loop_2^{ext}(X//G)$)

- the same objects as $Loop_2(X//G)$;
- A morphism

$$(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

  - $\alpha : \mathbb{R} \longrightarrow G$ smooth
  - $t \in \mathbb{R}$

$$\gamma'(s) = \gamma(s-t)\alpha(s-t).$$

S^1-action

Example ($Loop_1^{ext}(X//G)$)

- the same objects as $Loop_1(X//G)$;
- $(t,\alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ \downarrow t & & \downarrow \alpha & \nearrow f' & \\ S^1 & \xleftarrow[\pi']{} & P' & & \end{array}$$

Example ($Loop_2^{ext}(X//G)$)

- the same objects as $Loop_2(X//G)$;
- A morphism

$$(\alpha, t) : (\sigma, \gamma) \longrightarrow (\sigma', \gamma')$$

  - $\alpha : \mathbb{R} \longrightarrow G$ smooth
  - $t \in \mathbb{R}$

$$\gamma'(s) = \gamma(s-t)\alpha(s-t).$$

A skeleton of $Loop_2^{ext}(X//G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X//G)$.
- $L_g G = \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t')$.
- For any $(\gamma, t) \in L_g G \rtimes \mathbb{T}$, and $\delta \in \mathcal{L}_g X$, $\delta \cdot (\gamma, t) := (s \mapsto \delta(s+t) \cdot \gamma(s+t))$.

$$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X // L_g G \rtimes \mathbb{T} \text{ is a skeleton of } Loop_2^{ext}(X//G).$$

$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite. So we consider the subgroup consisting of constant loops

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

When G is a compact Lie group, $\Lambda_G(g)$ is a compact Lie group.

A skeleton of $Loop_2^{ext}(X//G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X//G)$.
- $L_g G = \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t')$.
- For any $(\gamma, t) \in L_g G \rtimes \mathbb{T}$, and $\delta \in \mathcal{L}_g X$, $\delta \cdot (\gamma, t) := (s \mapsto \delta(s+t) \cdot \gamma(s+t))$.

$$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X // L_g G \rtimes \mathbb{T} \text{ is a skeleton of } Loop_2^{ext}(X//G).$$

$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite. So we consider the subgroup consisting of constant loops

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

When G is a compact Lie group, $\Lambda_G(g)$ is a compact Lie group.

A skeleton of $Loop_2^{ext}(X//G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X//G)$.
- $L_g G = \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t')$.
- For any $(\gamma, t) \in L_g G \rtimes \mathbb{T}$, and $\delta \in \mathcal{L}_g X$, $\delta \cdot (\gamma, t) := (s \mapsto \delta(s+t) \cdot \gamma(s+t))$.

$$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X // L_g G \rtimes \mathbb{T} \text{ is a skeleton of } Loop_2^{ext}(X//G).$$

$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite. So we consider the subgroup consisting of constant loops

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

When G is a compact Lie group, $\Lambda_G(g)$ is a compact Lie group.

A skeleton of $Loop_2^{ext}(X//G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X//G)$.
- $L_g G = \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t')$.
- For any $(\gamma, t) \in L_g G \rtimes \mathbb{T}$, and $\delta \in \mathcal{L}_g X$, $\delta \cdot (\gamma, t) := (s \mapsto \delta(s+t) \cdot \gamma(s+t))$.

$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X // L_g G \rtimes \mathbb{T}$ is a skeleton of $Loop_2^{ext}(X//G)$.

$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite. So we consider the subgroup consisting of constant loops

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

When G is a compact Lie group, $\Lambda_G(g)$ is a compact Lie group.

The Groupoid with Good Data

A skeleton of $Loop_2^{ext}(X//G)$

- $\mathcal{L}_g X$: the space of objects (g,γ) in $Loop_2(X//G)$.
- $L_g G = \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R}\times G/(s+1,a)\sim(s,ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\gamma,t)\cdot(\gamma',t') := (s\mapsto \gamma(s)\gamma'(s+t), t+t')$.
- For any $(\gamma,t)\in L_g G\rtimes\mathbb{T}$, and $\delta\in\mathcal{L}_g X$, $\delta\cdot(\gamma,t) := (s\mapsto \delta(s+t)\cdot\gamma(s+t))$.

$\coprod_{g\in\pi_0 G/conj} \mathcal{L}_g X//L_g G\rtimes\mathbb{T}$ is a skeleton of $Loop_2^{ext}(X//G)$.

$L_g G\rtimes\mathbb{T}$ is an infinite dimensional topological group when G is not finite. So we consider the subgroup consisting of constant loops

$$\Lambda_G(g) = C_G(g)\times\mathbb{R}/\langle(g,-1)\rangle$$

When G is a compact Lie group, $\Lambda_G(g)$ is a compact Lie group.

The Groupoid with Good Data

A skeleton of $Loop_2^{ext}(X//G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X//G)$.
- $L_g G = \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t')$.
- For any $(\gamma, t) \in L_g G \rtimes \mathbb{T}$, and $\delta \in \mathcal{L}_g X$, $\delta \cdot (\gamma, t) := (s \mapsto \delta(s+t) \cdot \gamma(s+t))$.

$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X // L_g G \rtimes \mathbb{T}$ is a skeleton of $Loop_2^{ext}(X//G)$.

$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite.

So we consider the subgroup consisting of constant loops

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle(g, -1)\rangle$$

When G is a compact Lie group, $\Lambda_G(g)$ is a compact Lie group.

The Groupoid with Good Data

A skeleton of $Loop_2^{ext}(X//G)$

- $\mathcal{L}_g X$: the space of objects (g, γ) in $Loop_2(X//G)$.
- $L_g G = \{\gamma : \mathbb{R} \longrightarrow G | \gamma(s+1) = g^{-1}\gamma(s)g\}$, the gauge group of the principal $G-$bundle $P_g := \mathbb{R} \times G/(s+1, a) \sim (s, ga)$ over S^1;
- $L_g G \rtimes \mathbb{T}$: $(\gamma, t) \cdot (\gamma', t') := (s \mapsto \gamma(s)\gamma'(s+t), t+t')$.
- For any $(\gamma, t) \in L_g G \rtimes \mathbb{T}$, and $\delta \in \mathcal{L}_g X$, $\delta \cdot (\gamma, t) := (s \mapsto \delta(s+t) \cdot \gamma(s+t))$.

$\coprod_{g \in \pi_0 G/conj} \mathcal{L}_g X // L_g G \rtimes \mathbb{T}$ is a skeleton of $Loop_2^{ext}(X//G)$.

$L_g G \rtimes \mathbb{T}$ is an infinite dimensional topological group when G is not finite. So we consider the subgroup consisting of constant loops

$$\Lambda_G(g) = C_G(g) \times \mathbb{R}/\langle (g, -1) \rangle$$

When G is a compact Lie group, $\Lambda_G(g)$ is a compact Lie group.

Quasi-elliptic cohomology

$$\mathcal{L}(X//G) := \coprod_{g \in G^{tors}_{conj}} \mathcal{L}_g X // \Lambda_G(g)$$

$$\Lambda(X//G) := \coprod_{g \in G^{tors}_{conj}} X^g // \Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors}.

Quasi-elliptic cohomology

$$QEll(X//G) := K_{orb}(\Lambda(X//G)).$$

$QEll$ as equivariant $K-$theories

$$QEll(X//G) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X//G).$$

$$\mathcal{L}(X//G) := \coprod_{g \in G^{tors}_{conj}} \mathcal{L}_g X//\Lambda_G(g)$$

$$\Lambda(X//G) := \coprod_{g \in G^{tors}_{conj}} X^g//\Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors}.

Quasi-elliptic cohomology

$$QEll(X//G) := K_{orb}(\Lambda(X//G)).$$

$QEll$ as equivariant $K-$theories

$$QEll(X//G) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X//G).$$

Quasi-elliptic cohomology

$$\mathcal{L}(X//G) := \coprod_{g\in G^{tors}_{conj}} \mathcal{L}_g X//\Lambda_G(g)$$
$$\Lambda(X//G) := \coprod_{g\in G^{tors}_{conj}} X^g//\Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors}.

Quasi-elliptic cohomology

$$QEll(X//G) := K_{orb}(\Lambda(X//G)).$$

$QEll$ as equivariant $K-$theories

$$QEll(X//G) \cong \prod_{g\in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong K^*_{Tate}(X//G).$$

Quasi-elliptic cohomology

$$\mathcal{L}(X//G) := \coprod_{g\in G^{tors}_{conj}} \mathcal{L}_g X//\Lambda_G(g)$$

$$\Lambda(X//G) := \coprod_{g\in G^{tors}_{conj}} X^g//\Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors}.

Quasi-elliptic cohomology

$$QEll(X//G) := K_{orb}(\Lambda(X//G)).$$

QEll as equivariant $K-$theories

$$QEll(X//G) \cong \prod_{g\in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X//G).$$

Restriction

$\phi : X//G \longrightarrow Y//H \Longrightarrow \Lambda(\phi) : \Lambda(X//G) \longrightarrow \Lambda(Y//H)$

$$\begin{array}{ccc} QEll^*(Y//H) & \xrightarrow{\phi^*} & QEll^*(X//G) \\ \pi_{\phi(\tau)} \downarrow & & \pi_\tau \downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^\tau) \end{array}$$

Künneth Map

$K_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}((X \times Y)^{(\sigma,\tau)}) \cong K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)})$ where

$\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$: pullback of $\Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T} \xleftarrow{\pi} \Lambda_H(\tau)$.

$$QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) := \prod_{\sigma\in G^{tors}_{conj} \tau\in H^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau).$$

The Künneth map: $QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y)$.

Restriction

$\phi : X//G \longrightarrow Y//H \Longrightarrow \Lambda(\phi) : \Lambda(X//G) \longrightarrow \Lambda(Y//H)$

$$\begin{array}{ccc} QEll^*(Y//H) & \xrightarrow{\phi^*} & QEll^*(X//G) \\ \pi_{\phi(\tau)}\big\downarrow & & \pi_\tau\big\downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^\tau) \end{array}$$

Künneth Map

$K_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}((X \times Y)^{(\sigma,\tau)}) \cong K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)})$ where $\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$: pullback of $\Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T} \xleftarrow{\pi} \Lambda_H(\tau)$.

$$QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]}QEll^*_H(Y) := \prod_{\sigma\in G^{tors}_{conj}\tau\in H^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau).$$

The Künneth map: $QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]}QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y).$

Setting up the theory

Restriction

$\phi : X//G \longrightarrow Y//H \Longrightarrow \Lambda(\phi) : \Lambda(X//G) \longrightarrow \Lambda(Y//H)$

$$\begin{array}{ccc} QEll^*(Y//H) & \xrightarrow{\phi^*} & QEll^*(X//G) \\ \pi_{\phi(\tau)} \downarrow & & \pi_\tau \downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^\tau) \end{array}$$

Künneth Map

$K_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}((X \times Y)^{(\sigma,\tau)}) \cong K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)})$ where

$\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$: pullback of $\Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T} \xleftarrow{\pi} \Lambda_H(\tau)$.

$$QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]}QEll^*_H(Y) := \prod_{\sigma\in G^{tors}_{conj}\tau\in H^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau).$$

The Künneth map: $QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]}QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y)$.

Setting up the theory

Restriction

$\phi : X//G \longrightarrow Y//H \Longrightarrow \Lambda(\phi) : \Lambda(X//G) \longrightarrow \Lambda(Y//H)$

$$\begin{array}{ccc} QEll^*(Y//H) & \xrightarrow{\phi^*} & QEll^*(X//G) \\ \pi_{\phi(\tau)}\downarrow & & \pi_\tau\downarrow \\ K^*_{\Lambda_H(\phi(\tau))}(Y^{\phi(\tau)}) & \xrightarrow{\phi^*_\Lambda} & K^*_{\Lambda_G(\tau)}(X^\tau) \end{array}$$

Künneth Map

$K_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_H(\tau)}(Y^\tau) \longrightarrow K_{\Lambda_G(\sigma)\times_{\mathbb{T}}\Lambda_H(\tau)}((X \times Y)^{(\sigma,\tau)}) \cong K_{\Lambda_{G\times H}(\sigma,\tau)}((X \times Y)^{(\sigma,\tau)})$ where

$\Lambda_{G\times H}(\sigma,\tau) \cong \Lambda_G(\sigma) \times_{\mathbb{T}} \Lambda_H(\tau)$: pullback of $\Lambda_G(\sigma) \xrightarrow{\pi} \mathbb{T} \xleftarrow{\pi} \Lambda_H(\tau)$.

$$QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) := \prod_{\sigma\in G^{tors}_{conj}\tau\in H^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \otimes_{\mathbb{Z}[q^\pm]} K^*_{\Lambda_H(\tau)}(Y^\tau).$$

The Künneth map: $QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^\pm]} QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y)$.

Setting up the theory

The Change of Group Isomorphism

- H: closed subgroup of G;
- X: H-space;
- $\phi : H \longrightarrow G$ is the inclusion.

Theorem

The change-of-group map ρ_H^G is an isomorphism.

$$\rho_H^G : QEll_G^*(G \times_H X) \xrightarrow{\phi^*} QEll_H^*(G \times_H X) \xrightarrow{i^*} QEll_H^*(X)$$

- ϕ^*: *the restriction map*
- $i : X \longrightarrow G \times_H X$: $i(x) = [e, x]$.

Induced map

$$\mathcal{I}_H^G : QEll(X//H) \xrightarrow{\cong} QEll((G \times_H X)//G) \longrightarrow QEll(X//G)$$

- the first map: the change of group isomorphism
- the second: the finite covering $\Lambda(G \times_H X//G) \longrightarrow \Lambda(X//G)$
  obj $(\sigma, [g, x]) \mapsto (\sigma, gx)$; mor $([g', t], (\sigma, [g, x])) \mapsto ([g', t], (gx, \sigma))$.

Setting up the theory

The Change of Group Isomorphism

- H: closed subgroup of G;
- X: H-space;
- $\phi : H \longrightarrow G$ is the inclusion.

Theorem

The change-of-group map ρ_H^G is an isomorphism.

$$\rho_H^G : QEll_G^*(G \times_H X) \xrightarrow{\phi^*} QEll_H^*(G \times_H X) \xrightarrow{i^*} QEll_H^*(X)$$

- ϕ^*: *the restriction map*
- $i : X \longrightarrow G \times_H X$: $i(x) = [e, x]$.

Induced map

$$\mathcal{I}_H^G : QEll(X//H) \xrightarrow{\cong} QEll((G \times_H X)//G) \longrightarrow QEll(X//G)$$

- the first map: the change of group isomorphism
- the second: the finite covering $\Lambda(G \times_H X//G) \longrightarrow \Lambda(X//G)$
  obj $(\sigma, [g, x]) \mapsto (\sigma, gx)$; mor $([g', t], (\sigma, [g, x])) \mapsto ([g', t], (gx, \sigma))$.

Setting up the theory

The Change of Group Isomorphism

- H: closed subgroup of G;
- X: H-space;
- $\phi : H \longrightarrow G$ is the inclusion.

Theorem

The change-of-group map ρ^G_H is an isomorphism.

$$\rho^G_H : QEll^*_G(G \times_H X) \xrightarrow{\phi^*} QEll^*_H(G \times_H X) \xrightarrow{i^*} QEll^*_H(X)$$

- ϕ^*: *the restriction map*
- $i : X \longrightarrow G \times_H X$: $i(x) = [e, x]$.

Induced map

$$\mathcal{I}^G_H : QEll(X//H) \xrightarrow{\cong} QEll((G \times_H X)//G) \longrightarrow QEll(X//G)$$

- the first map: the change of group isomorphism
- the second: the finite covering $\Lambda(G \times_H X//G) \longrightarrow \Lambda(X//G)$ obj $(\sigma, [g, x]) \mapsto (\sigma, gx)$; mor $([g', t], (\sigma, [g, x])) \mapsto ([g', t], (gx, \sigma))$.

Power Operation

Quasi-elliptic cohomology has power operations, which gives it the structure of an "$H_\infty-$ring theory" [Ganter 06].

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X//G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

"Elliptic" Power Operation [Ganter] [Ando]

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = K^*_{Tate}(X//G)$$

There exists

$$\mathbb{P}_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

extends uniquely to the stringy power operation

$$P^{string}_n : K_{Tate}(X//G) \longrightarrow K_{Tate}(X^{\times n}//G\wr\Sigma_n)$$

which is elliptic.

Power Operation

Quasi-elliptic cohomology has power operations, which gives it the structure of an "H_∞−ring theory" [Ganter 06].

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X//G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

"Elliptic" Power Operation [Ganter] [Ando]

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = K^*_{Tate}(X//G)$$

There exists

$$\mathbb{P}_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

extends uniquely to the stringy power operation

$$P^{string}_n : K_{Tate}(X//G) \longrightarrow K_{Tate}(X^{\times n}//G \wr \Sigma_n)$$

which is elliptic.

Power Operation

Quasi-elliptic cohomology has power operations, which gives it the structure of an "H_∞−ring theory" [Ganter 06].

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X//G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

"Elliptic" Power Operation [Ganter] [Ando]

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = K^*_{Tate}(X//G)$$

There exists

$$\mathbb{P}_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

extends uniquely to the stringy power operation

$$P^{string}_n : K_{Tate}(X//G) \longrightarrow K_{Tate}(X^{\times n}//G \wr \Sigma_n)$$

which is elliptic.

The Construction of $\mathbb{P}_n$

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)} :$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}).$$

Each $\mathbb{P}_{(\underline{g},\sigma)}$ is constructed as the composition:

$$QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\Lambda}_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

$$\xrightarrow{\boxtimes} K_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}),$$

The Construction of $\mathbb{P}_n$

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)} :$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}).$$

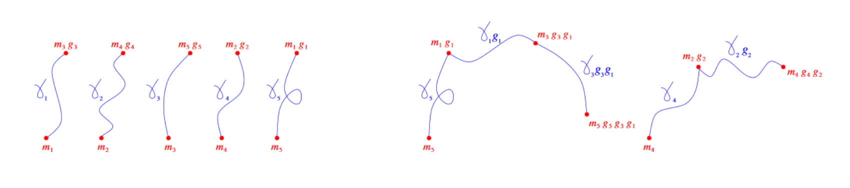
Each $\mathbb{P}_{(\underline{g},\sigma)}$ is constructed as the composition:

$$QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\Lambda}_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

$$\xrightarrow{\boxtimes} K_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}),$$

$\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ and $\prod\limits_{k}\prod\limits_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}}X$ are $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$equivariant homeomorphic.

Example $(\mathcal{L}_{(g_1,\cdots g_5,(135)(24))}(X^{\times 5})$ and ${}_3\mathcal{L}_{g_5g_3g_1}(X)\times{}_2\mathcal{L}_{g_4g_2}(X))$



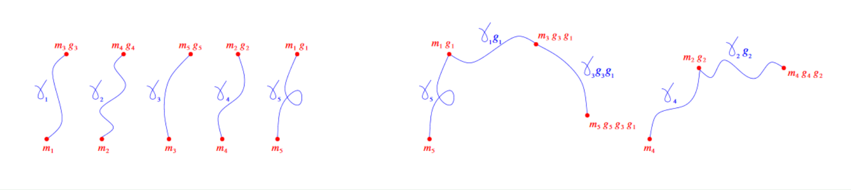
This picture is from [Ganter].

$f_{(\underline{g},\sigma)}$ is the restriction of this homeomorphism to the constant loops.

the functor $f_{(\underline{g},\sigma)}$: the KEY isomorphism

$\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ and $\prod_{k} \prod_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}} X$ are $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$equivariant homeomorphic.

Example ($\mathcal{L}_{(g_1,\cdots g_5,(135)(24))}(X^{\times 5})$ and ${}_3\mathcal{L}_{g_5g_3g_1}(X) \times {}_2\mathcal{L}_{g_4g_2}(X)$)



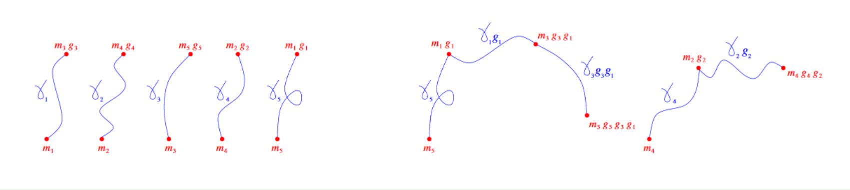
This picture is from [Ganter].

$f_{(\underline{g},\sigma)}$ is the restriction of this homeomorphism to the constant loops.

the functor $f_{(\underline{g},\sigma)}$: the KEY isomorphism

$\mathcal{L}_{(\underline{g},\sigma)}(X^{\times n})$ and $\prod\limits_{k}\prod\limits_{(i_1,\cdots i_k)} {}_k\mathcal{L}_{g_{i_k}\cdots g_{i_1}}X$ are $\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)-$equivariant homeomorphic.

Example $(\mathcal{L}_{(g_1,\cdots g_5,(135)(24))}(X^{\times 5})$ and ${}_3\mathcal{L}_{g_5g_3g_1}(X)\times {}_2\mathcal{L}_{g_4g_2}(X))$



This picture is from [Ganter].

$f_{(\underline{g},\sigma)}$ is the restriction of this homeomorphism to the constant loops.

$$C_G(g, g') := \{x \in G | gx = xg'\}.$$

$$\Lambda^k_G(g, g') := C_G(g, g') \times \mathbb{R}/(x, t) \sim (gx, t - k).$$

Note

$$\Lambda^1_G(g, g) = \Lambda_G(g).$$

A groupoid equivalent to $\Lambda(X//G)$

- objects $\coprod_{g \in G^{tors}} X^g$;
- morphisms $\coprod_{g, g' \in G^{tors}} \Lambda^1_G(g, g') \times X^g$

  Explicitly, (α, x) maps $x \in X^g$ to $\alpha \cdot x \in X^{g'}$.

Let's use the same symbol $\Lambda(X//G)$ to denote it.

Construction and Notation

$$C_G(g, g') := \{x \in G | gx = xg'\}.$$

$$\Lambda^k_G(g, g') := C_G(g, g') \times \mathbb{R}/(x, t) \sim (gx, t - k).$$

Note

$$\Lambda^1_G(g, g) = \Lambda_G(g).$$

A groupoid equivalent to $\Lambda(X//G)$

- objects $\coprod_{g \in G^{tors}} X^g$;
- morphisms $\coprod_{g, g' \in G^{tors}} \Lambda^1_G(g, g') \times X^g$

  Explicitly, (α, x) maps $x \in X^g$ to $\alpha \cdot x \in X^{g'}$.

Let's use the same symbol $\Lambda(X//G)$ to denote it.

$$C_G(g, g') := \{x \in G | gx = xg'\}.$$

$$\Lambda_G^k(g, g') := C_G(g, g') \times \mathbb{R}/(x, t) \sim (gx, t - k).$$

Note

$$\Lambda_G^1(g, g) = \Lambda_G(g).$$

A groupoid equivalent to $\Lambda(X//G)$

- objects $\coprod_{g \in G^{tors}} X^g$;
- morphisms $\coprod_{g, g' \in G^{tors}} \Lambda_G^1(g, g') \times X^g$

  Explicitly, (α, x) maps $x \in X^g$ to $\alpha \cdot x \in X^{g'}$.

Let's use the same symbol $\Lambda(X//G)$ to denote it.

For $(\underline{g}, \sigma) \in G \wr \Sigma_n$, let $\Lambda_{(\underline{g},\sigma)}(X)$ denote the groupoid with

- objects: points in $\coprod\limits_{k} \coprod\limits_{(i_1,\cdots i_k)} X^{g_{i_k}\cdots g_{i_1}}$
  where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ;
- morphisms: $\coprod\limits_{k} \coprod\limits_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$,
  Explicitly, (α, x) maps x to $\alpha \cdot x$.

Let

$$U : \Lambda_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda(X//G)$$

denote the functor sending

- an object $x \in X^{g_{i_k}\cdots g_{i_1}} \Longrightarrow x \in X^{g_{i_k}\cdots g_{i_1}}$;
- a morphism $(\alpha, x) \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}} \Longrightarrow$
  $(\alpha, x) \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$.

Note: U may not be the identity map.

the Functor U: Draw necessary information

For $(\underline{g}, \sigma) \in G \wr \Sigma_n$, let $\Lambda_{(\underline{g},\sigma)}(X)$ denote the groupoid with

- objects: points in $\coprod_k \coprod_{(i_1,\cdots i_k)} X^{g_{i_k}\cdots g_{i_1}}$
  where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ;
- morphisms: $\coprod_k \coprod_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$,
  Explicitly, (α, x) maps x to $\alpha \cdot x$.

Let

$$U : \Lambda_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda(X//G)$$

denote the functor sending

- an object $x \in X^{g_{i_k}\cdots g_{i_1}} \Longrightarrow x \in X^{g_{i_k}\cdots g_{i_1}}$;
- a morphism $(\alpha, x) \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}} \Longrightarrow$
  $(\alpha, x) \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$.

Note: U may not be the identity map.

For $(\underline{g}, \sigma) \in G \wr \Sigma_n$, let $\Lambda_{(\underline{g},\sigma)}(X)$ denote the groupoid with

- objects: points in $\coprod_{k} \coprod_{(i_1, \cdots i_k)} X^{g_{i_k} \cdots g_{i_1}}$
  where $(i_1, \cdots i_k)$ goes over all the $k-$cycles of σ;
- morphisms: $\coprod_{k} \coprod_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda_G(g_{i_k} \cdots g_{i_1}, g_{j_k} \cdots g_{j_1}) \times X^{g_{i_k} \cdots g_{i_1}}$,
  Explicitly, (α, x) maps x to $\alpha \cdot x$.

Let

$$U : \Lambda_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda(X//G)$$

denote the functor sending

- an object $x \in X^{g_{i_k} \cdots g_{i_1}} \Longrightarrow x \in X^{g_{i_k} \cdots g_{i_1}}$;
- a morphism $(\alpha, x) \in \Lambda_G(g_{i_k} \cdots g_{i_1}, g_{j_k} \cdots g_{j_1}) \times X^{g_{i_k} \cdots g_{i_1}} \Longrightarrow$
  $(\alpha, x) \in \Lambda_G(g_{i_k} \cdots g_{i_1}, g_{j_k} \cdots g_{j_1}) \times X^{g_{i_k} \cdots g_{i_1}}$.

Note: U may not be the identity map.

Let $\Lambda^{var}_{(\underline{g},\sigma)}(X)$ be the groupoid with

- the same objects as $\Lambda_{(\underline{g},\sigma)}(X)$
- morphisms: $\coprod\limits_{k}\ \coprod\limits_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})\times X^{g_{i_k}\cdots g_{i_1}}$, where $(i_1,\cdots i_k)$ and $(j_1,\cdots j_k)$ go over all the $k-$cycles of σ.

The functor

$$(\ )^{\Lambda}_k : \Lambda^{var}_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda_{(\underline{g},\sigma)}(X)$$

- identity on objects
- sends each $[g,t]\in\Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$ to $[g,\frac{t}{k}]\in\Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$.

Let $\Lambda^{var}_{(\underline{g},\sigma)}(X)$ be the groupoid with

- the same objects as $\Lambda_{(\underline{g},\sigma)}(X)$
- morphisms: $\coprod\limits_{k} \coprod\limits_{(i_1,\cdots i_k),(j_1,\cdots j_k)} \Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1}) \times X^{g_{i_k}\cdots g_{i_1}}$, where $(i_1,\cdots i_k)$ and $(j_1,\cdots j_k)$ go over all the $k-$cycles of σ.

The functor

$$(\ )^{\Lambda}_k : \Lambda^{var}_{(\underline{g},\sigma)}(X) \longrightarrow \Lambda_{(\underline{g},\sigma)}(X)$$

- identity on objects
- sends each $[g,t] \in \Lambda^k_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$ to $[g,\frac{t}{k}] \in \Lambda_G(g_{i_k}\cdots g_{i_1}, g_{j_k}\cdots g_{j_1})$.

Examples

Example ($n = 2$ and $G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. $\sigma \in \Sigma_n$. $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.

$$QEll(X//\Sigma_2) \cong K(X)[q^{\pm}][1,s]/(s^2-1) \times K(X)[q^{\pm}][y]/(y^2-q)$$

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

Transfer Ideal for QEll

$$\mathcal{I}^{QEll}_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[\mathcal{I}^{\Sigma_N}_{\Sigma_i\times\Sigma_j} : QEll(\mathrm{pt}//\Sigma_i \times \Sigma_j) \longrightarrow QEll(\mathrm{pt}//\Sigma_N)]$$

Example ($n = 2$ and $X =$pt)

$\mathcal{I}^{QEll}_{tr} = Ind^{\Lambda_{\Sigma_2}(1)}_{\Lambda_{\Sigma_1\times\Sigma_1}(1)} K_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}(\mathrm{pt})$. $\mathbb{P}_2(q) = (q^2, q^{\frac{1}{2}})$.

$$QEll(\mathrm{pt}//\Sigma_2)/\mathcal{I}^{QEll}_{tr} = K_{\Lambda_{\Sigma_2}(1)}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr} \times K_{\Lambda_{\Sigma_2}(12)}(\mathrm{pt})$$

$$\cong \mathbb{Z}[q^{\pm}][q']/(q'-q^2) \times \mathbb{Z}[q^{\pm}][q']/(q'^2-q).$$

Examples

Example ($n = 2$ and $G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. $\sigma \in \Sigma_n$. $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.

$$QEll(X//\Sigma_2) \cong K(X)[q^{\pm}][1,s]/(s^2-1) \times K(X)[q^{\pm}][y]/(y^2-q)$$

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

Transfer Ideal for QEll

$$\mathcal{I}^{QEll}_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[\mathcal{I}^{\Sigma_N}_{\Sigma_i \times \Sigma_j} : QEll(\mathrm{pt}//\Sigma_i \times \Sigma_j) \longrightarrow QEll(\mathrm{pt}//\Sigma_N)]$$

Example ($n = 2$ and $X =$pt)

$\mathcal{I}^{QEll}_{tr} = Ind^{\Lambda_{\Sigma_2}(1)}_{\Lambda_{\Sigma_1\times\Sigma_1}(1)} K_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}(\mathrm{pt})$. $\mathbb{P}_2(q) = (q^2, q^{\frac{1}{2}})$.

$$QEll(\mathrm{pt}//\Sigma_2)/\mathcal{I}^{QEll}_{tr} = K_{\Lambda_{\Sigma_2}(1)}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr} \times K_{\Lambda_{\Sigma_2}(12)}(\mathrm{pt})$$

$$\cong \mathbb{Z}[q^{\pm}][q']/(q'-q^2) \times \mathbb{Z}[q^{\pm}][q']/(q'^2-q).$$

Examples

Example ($n=2$ and $G=e$)

$QEll^*_G(X)=K^*_{\mathbb{T}}(X)$. $\sigma\in\Sigma_n$. $\mathbb{P}_{(\underline{1},\sigma)}(x)=\boxtimes_k\boxtimes_{(i_1,\cdots i_k)}(x)_k$.

$$QEll(X//\Sigma_2)\cong K(X)[q^{\pm}][1,s]/(s^2-1)\times K(X)[q^{\pm}][y]/(y^2-q)$$

$$\mathbb{P}_2(x)=(\mathbb{P}_{(\underline{1},(1)(1))}(x),\mathbb{P}_{(\underline{1},(12))}(x))=(x\boxtimes x,(x)_2).$$

Transfer Ideal for QEll

$$\mathcal{I}^{QEll}_{tr}:=\sum_{\substack{i+j=N,\\ N>j>0}}\mathrm{Image}[\mathcal{I}^{\Sigma_N}_{\Sigma_i\times\Sigma_j}:QEll(\mathrm{pt}//\Sigma_i\times\Sigma_j)\longrightarrow QEll(\mathrm{pt}//\Sigma_N)]$$

Example ($n=2$ and $X=$pt)

$\mathcal{I}^{QEll}_{tr}=Ind^{\Lambda_{\Sigma_2}(1)}_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}K_{\Lambda_{\Sigma_1\times\Sigma_1}(1)}(\mathrm{pt})$. $\mathbb{P}_2(q)=(q^2,q^{\frac{1}{2}})$.

$$\begin{aligned}QEll(\mathrm{pt}//\Sigma_2)/\mathcal{I}^{QEll}_{tr}&=K_{\Lambda_{\Sigma_2}(1)}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr}\times K_{\Lambda_{\Sigma_2}(12)}(\mathrm{pt})\\&\cong\mathbb{Z}[q^{\pm}][q']/(q'-q^2)\times\mathbb{Z}[q^{\pm}][q']/(q'^2-q).\end{aligned}$$

Finite Subgroups of Tate Curve

Theorem (H.)

$$QEll(pt//\Sigma_N)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e \rangle,$$

where $\mathcal{I}_{tr}^{QEll}$ is the transfer ideal and q' is the image of q under the power operation $\mathbb{P}_N$. The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Theorem (H.)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt//\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e} \rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q_s' is the image of q under the stringy power operation P_N^{string}, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Finite Subgroups of Tate Curve

Theorem (H.)

$$QEll(pt//\Sigma_N)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e \rangle,$$

where $\mathcal{I}_{tr}^{QEll}$ is the transfer ideal and q' is the image of q under the power operation $\mathbb{P}_N$. The product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Theorem (H.)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt//\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e} \rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q'_s is the image of q under the stringy power operation P_N^{string}, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

Sketch of the Proof

Case I:

$\sigma \in \Sigma_N$ has cycles with different length. Then $\sigma \in \Sigma_r \times \Sigma_{N-r}$ such that all the cycles of of the same length are either in Σ_r or Σ_{N-r}.

$$\Lambda_{\Sigma_N}(\sigma) = \Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma) \Longrightarrow Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma)} \text{ is the identity map.}$$

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo } Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma)} K_{\Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma)}(\mathrm{pt}) \text{ is trivial.}$$

Case II:

$\sigma \in \Sigma_N$ has d $e-$cycles. $\Lambda_{\Sigma_N}(\sigma) \cong \Lambda_{\Sigma_e}(12\cdots e) \wr_{\mathbb{T}} \Sigma_d$.
$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \cong R\Lambda_{\Sigma_N}(\sigma)$ has a $\mathbb{Z}[q^{\pm}]-$basis

$$\{Ind^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_{(d)}} \left((q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d_1} \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} (q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q^{\pm}]} d_r} \right)^{\sim} \otimes D_\tau$$

$$\mid (d) = (d_1, d_2, \cdots d_r) \text{ is a partition of } d.\ a_1 < a_2 < \cdots < a_r$$

are in $\{0, 1, \cdots e-1\}$. And $\tau \in R\Sigma_{(d)}$ is irreducible.$\}$

Sketch of the Proof

Case I:

$\sigma \in \Sigma_N$ has cycles with different length. Then $\sigma \in \Sigma_r \times \Sigma_{N-r}$ such that all the cycles of of the same length are either in Σ_r or Σ_{N-r}.

$$\Lambda_{\Sigma_N}(\sigma) = \Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma) \Longrightarrow Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma)} \text{ is the identity map.}$$

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo } Ind^{\Lambda_{\Sigma_N}(\sigma)}_{\Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma)} K_{\Lambda_{\Sigma_r \times \Sigma_{N-r}}(\sigma)}(\mathrm{pt}) \text{ is trivial.}$$

Case II:

$\sigma \in \Sigma_N$ has d $e-$cycles. $\Lambda_{\Sigma_N}(\sigma) \cong \Lambda_{\Sigma_e}(12 \cdots e) \wr_{\mathbb{T}} \Sigma_d$.
$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \cong R\Lambda_{\Sigma_N}(\sigma)$ has a $\mathbb{Z}[q^{\pm}]-$basis

$$\{Ind^{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_d}_{\Lambda_{\Sigma_e}(12\cdots e)\wr_{\mathbb{T}}\Sigma_{(d)}} \left((q^{\frac{a_1}{e}})^{\otimes_{\mathbb{Z}[q^\pm]} d_1} \otimes_{\mathbb{Z}[q^\pm]} \cdots \otimes_{\mathbb{Z}[q^\pm]} (q^{\frac{a_r}{e}})^{\otimes_{\mathbb{Z}[q^\pm]} d_r} \right)^{\sim} \otimes D_\tau$$

$$\mid (d) = (d_1, d_2, \cdots d_r) \text{ is a partition of } d.\ a_1 < a_2 < \cdots < a_r$$

are in $\{0, 1, \cdots e-1\}$. And $\tau \in R\Sigma_{(d)}$ is irreducible.$\}$

Sketch of the Proof

Continue with Case II

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt}) \text{ modulo the image of the transfer}$$

is a $\mathbb{Z}[q^{\pm}]-$module generated by

$$\{\mathbb{P}_\sigma(q)^{\otimes a} \mid a = 0, 1, \cdots e-1\}.$$

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\text{pt}) \text{ modulo the image of the transfer } \cong \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e\rangle.$$

$$\Longrightarrow QEll(\text{pt}//\Sigma_N)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e\rangle.$$

Corollary

$$QEll^*(X//G) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X//G)$$

$$\mathbb{P}_N \sim P_N^{string}$$

$$\Longrightarrow K_{Tate}(\text{pt}//\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^e\rangle,$$

Sketch of the Proof

Continue with Case II

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo the image of the transfer}$$

is a $\mathbb{Z}[q^{\pm}]-$module generated by

$$\{\mathbb{P}_\sigma(q)^{\otimes a} \mid a = 0, 1, \cdots e-1\}.$$

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo the image of the transfer } \cong \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e\rangle.$$

$$\Longrightarrow QEll(\mathrm{pt}//\Sigma_N)/\mathcal{I}^{QEll}_{tr} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e\rangle.$$

Corollary

$$QEll^*(X//G)\otimes_{\mathbb{Z}[q^{\pm}]}\mathbb{Z}((q)) \cong K^*_{Tate}(X//G)$$

$$\mathbb{P}_N \sim P_N^{string}$$

$$\Longrightarrow K_{Tate}(\mathrm{pt}//\Sigma_N)/I^{Tate}_{tr} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle,$$

Sketch of the Proof

Continue with Case II

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo the image of the transfer}$$

is a $\mathbb{Z}[q^{\pm}]-$module generated by

$$\{\mathbb{P}_\sigma(q)^{\otimes a} \mid a = 0, 1, \cdots e - 1\}.$$

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo the image of the transfer } \cong \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e \rangle.$$

$$\Longrightarrow QEll(\mathrm{pt}//\Sigma_N)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e \rangle.$$

Corollary

$$QEll^*(X//G) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X//G)$$

$$\mathbb{P}_N \sim P_N^{string}$$

$$\Longrightarrow K_{Tate}(\mathrm{pt}//\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e} \rangle,$$

Continue with Case II

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo the image of the transfer}$$

is a $\mathbb{Z}[q^{\pm}]-$module generated by

$$\{\mathbb{P}_\sigma(q)^{\otimes a} \mid a = 0, 1, \cdots e-1\}.$$

$$K_{\Lambda_{\Sigma_N}(\sigma)}(\mathrm{pt}) \text{ modulo the image of the transfer } \cong \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e \rangle.$$

$$\Longrightarrow QEll(\mathrm{pt}//\Sigma_N)/\mathcal{I}_{tr}^{QEll} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e \rangle.$$

Corollary

$$QEll^*(X//G) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X//G)$$

$$\mathbb{P}_N \sim P_N^{string}$$

$$\Longrightarrow K_{Tate}(\mathrm{pt}//\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[{q'_s}^{\pm}]/\langle q^d - {q'_s}^e \rangle,$$

We can define an additive operation

$$\begin{aligned}\overline{P}_N : & QEll_G(X) \xrightarrow{\mathbb{P}_N} QEll_{G\wr\Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G\times\Sigma_N}(X^{\times N}) \\ & \xrightarrow{diag^*} QEll_{G\times\Sigma_N}(X) \cong QEll_G(X)\otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt}) \\ & \longrightarrow QEll_G(X)\otimes_{\mathbb{Z}[q^{\pm}]} QEll_{\Sigma_N}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr} \\ & \cong QEll_G(X)\otimes_{\mathbb{Z}[q^{\pm}]} \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e\rangle,\end{aligned}$$

It is the Adams operation of quasi-elliptic cohomology. It extends uniquely to an additive operation

$$\overline{P^{string}}_n : K_{Tate}(X//G) \longrightarrow K_{Tate}(X//G)\otimes_{\mathbb{Z}((q))}\left(K_{Tate}(\mathrm{pt}//\Sigma_N)/I^{Tate}_{tr}\right).$$

Taking the trace of $\overline{P^{string}}_n(x)$, it equals $nT_n(x)$ with T_n the Hecke operator of Tate K-theory defined by the stringy power operation P^{string}_n, as shown in [Ganter].

We can define an additive operation

$$\begin{aligned}\overline{P}_N : & QEll_G(X) \xrightarrow{\mathbb{P}_N} QEll_{G\wr\Sigma_N}(X^{\times N}) \xrightarrow{res} QEll_{G\times\Sigma_N}(X^{\times N}) \\ & \xrightarrow{diag^*} QEll_{G\times\Sigma_N}(X) \cong QEll_G(X)\otimes_{\mathbb{Z}[q^\pm]} QEll_{\Sigma_N}(\mathrm{pt}) \\ & \longrightarrow QEll_G(X)\otimes_{\mathbb{Z}[q^\pm]} QEll_{\Sigma_N}(\mathrm{pt})/\mathcal{I}^{QEll}_{tr} \\ & \cong QEll_G(X)\otimes_{\mathbb{Z}[q^\pm]} \prod_{N=de} \mathbb{Z}[q^\pm][q'^\pm]/\langle q^d - q'^e\rangle,\end{aligned}$$

It is the Adams operation of quasi-elliptic cohomology. It extends uniquely to an additive operation

$$\overline{P^{string}}_n : K_{Tate}(X//G) \longrightarrow K_{Tate}(X//G)\otimes_{\mathbb{Z}((q))}\left(K_{Tate}(\mathrm{pt}//\Sigma_N)/I^{Tate}_{tr}\right).$$

Taking the trace of $\overline{P^{string}}_n(x)$, it equals $nT_n(x)$ with T_n the Hecke operator of Tate K-theory defined by the stringy power operation P^{string}_n, as shown in [Ganter].

Spectra

$G-$spectra

We construct a space $QEll_{G,n}$ for each G and each n representing $QEll^n_G(-)$ in the sense

$$\pi_0(QEll_{G,n}) = QEll^n_G(S^0).$$

Orthogonal $G-$spectra

We construct an orthogonal $G-$spectra E, which is a $\mathcal{I}_G-$FSP. For each faithful $G-$representation V, E weakly represents $QEll^V_G(-)$ in the sense

$$\pi_k(E(G,V)) = QEll^V_G(S^k), \text{ for each } k.$$

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not a trivial.
$\Longrightarrow \{E(G,-)\}$ cannot arise from an orthogonal spectrum.

Spectra

$G-$spectra

We construct a space $QEll_{G,n}$ for each G and each n representing $QEll^n_G(-)$ in the sense

$$\pi_0(QEll_{G,n}) = QEll^n_G(S^0).$$

Orthogonal $G-$spectra

We construct an orthogonal $G-$spectra E, which is a $\mathcal{I}_G-$FSP. For each faithful $G-$representation V, E weakly represents $QEll^V_G(-)$ in the sense

$$\pi_k(E(G,V)) = QEll^V_G(S^k), \text{ for each } k.$$

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not a trivial.
$\Longrightarrow \{E(G,-)\}$ cannot arise from an orthogonal spectrum.

Spectra

$G-$spectra

We construct a space $QEll_{G,n}$ for each G and each n representing $QEll^n_G(-)$ in the sense

$$\pi_0(QEll_{G,n}) = QEll^n_G(S^0).$$

Orthogonal $G-$spectra

We construct an orthogonal $G-$spectra E, which is a $\mathcal{I}_G-$FSP. For each faithful $G-$representation V, E weakly represents $QEll^V_G(-)$ in the sense

$$\pi_k(E(G,V)) = QEll^V_G(S^k), \text{ for each } k.$$

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not a trivial.

$\Longrightarrow \{E(G,-)\}$ cannot arise from an orthogonal spectrum.

Spectra

$G-$spectra

We construct a space $QEll_{G,n}$ for each G and each n representing $QEll_G^n(-)$ in the sense

$$\pi_0(QEll_{G,n}) = QEll_G^n(S^0).$$

Orthogonal $G-$spectra

We construct an orthogonal $G-$spectra E, which is a $\mathcal{I}_G-$FSP. For each faithful $G-$representation V, E weakly represents $QEll_G^V(-)$ in the sense

$$\pi_k(E(G,V)) = QEll_G^V(S^k), \text{ for each } k.$$

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not a trivial.
$\Longrightarrow \{E(G,-)\}$ cannot arise from an orthogonal spectrum.

Sketch of the Construction

- V : $G-$representation
- $Sym(V) := \bigoplus_{n\geq 0} Sym^n(V)$: the total symmetric power.
- $S(G,V)_g := Sym(V) \setminus Sym(V)^g$.
- KU: the global complex $K-$spectrum;
- $(V)_g$: a specific $\Lambda_G(g)-$representation.

$F_g(G,V) := \mathrm{Map}_{\mathbb{R}}(S^{(V)_g}, KU((V)_g \oplus V^g))$
$E_g(G,V) := \{t_1 a + t_2 b \in F_g(G,V) * S(G,V)_g \mid \|b\| \leqslant t_2\}/\{t_1 c_0 + t_2 b\}.$

$$E(G,V) := \prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G,V))$$

- V : $G-$representation
- $Sym(V) := \bigoplus_{n\geq 0} Sym^n(V)$: the total symmetric power.
- $S(G,V)_g := Sym(V) \setminus Sym(V)^g$.
- KU: the global complex $K-$spectrum;
- $(V)_g$: a specific $\Lambda_G(g)-$representation.

$F_g(G,V) := \mathrm{Map}_{\mathbb{R}}(S^{(V)_g}, KU((V)_g \oplus V^g))$

$E_g(G,V) := \{t_1 a + t_2 b \in F_g(G,V) * S(G,V)_g | \|b\| \leqslant t_2\}/\{t_1 c_0 + t_2 b\}.$

$$E(G,V) := \prod_{g\in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G,V))$$

Sketch of the Construction

- V : $G-$representation
- $Sym(V) := \bigoplus_{n\geq 0} Sym^n(V)$: the total symmetric power.
- $S(G,V)_g := Sym(V) \setminus Sym(V)^g$.
- KU: the global complex $K-$spectrum;
- $(V)_g$: a specific $\Lambda_G(g)-$representation.

$F_g(G,V) := \mathrm{Map}_{\mathbb{R}}(S^{(V)_g}, KU((V)_g \oplus V^g))$
$E_g(G,V) := \{t_1 a + t_2 b \in F_g(G,V) * S(G,V)_g \mid \|b\| \leqslant t_2\}/\{t_1 c_0 + t_2 b\}$.

$$E(G,V) := \prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G,V))$$

- V : $G-$representation
- $Sym(V) := \bigoplus_{n\geq 0} Sym^n(V)$: the total symmetric power.
- $S(G, V)_g := Sym(V) \setminus Sym(V)^g$.
- KU: the global complex $K-$spectrum;
- $(V)_g$: a specific $\Lambda_G(g)-$representation.

$F_g(G, V) := \mathrm{Map}_{\mathbb{R}}(S^{(V)_g}, KU((V)_g \oplus V^g))$
$E_g(G, V) := \{t_1 a + t_2 b \in F_g(G, V) * S(G, V)_g | \|b\| \leqslant t_2\}/\{t_1 c_0 + t_2 b\}$.

$$E(G, V) := \prod_{g \in G^{tors}_{conj}} \mathrm{Map}_{C_G(g)}(G, E_g(G, V))$$

a new Global Homotopy Theory

A category D_0 larger than $\mathbb{L}$

- objects: (G, V, ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations $\rho : G \longrightarrow O(V)$,
- morphism: $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$
  - $\phi_2 : V \longrightarrow W$ a linear isometric embedding
  - $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow G$ group homomorphism

$$\begin{array}{ccc} G & \xrightarrow{\rho} & O(V) \\ \big\uparrow{\scriptstyle \phi_1} & & \big\downarrow{\scriptstyle \phi_{2*}} \\ \tau^{-1}(O(\phi_2(V))) & \xrightarrow{\tau} & O(W) \end{array}$$

The Category of D_0-Spaces

A D_0-space is a continuous functor $X : D_0 \longrightarrow T$ to the category of compactly generated weak Hausdorff spaces. A morphism of D_0-spaces is a natural transformation.

A category D_0 larger than $\mathbb{L}$

- objects: (G, V, ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations $\rho : G \longrightarrow O(V)$,
- morphism: $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$
  - $\phi_2 : V \longrightarrow W$ a linear isometric embedding
  - $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow G$ group homomorphism

$$\begin{array}{ccc} G & \xrightarrow{\ \rho\ } & O(V) \\ {\scriptstyle\phi_1}\uparrow & & \downarrow{\scriptstyle\phi_{2*}} \\ \tau^{-1}(O(\phi_2(V))) & \xrightarrow{\ \tau\ } & O(W) \end{array}$$

The Category of D_0-Spaces

A D_0-space is a continuous functor $X : D_0 \longrightarrow T$ to the category of compactly generated weak Hausdorff spaces. A morphism of D_0-spaces is a natural transformation.

A category D_0 larger than $\mathbb{L}$

- objects: (G, V, ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations $\rho : G \longrightarrow O(V)$,
- morphism: $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$
  - $\phi_2 : V \longrightarrow W$ a linear isometric embedding
  - $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow G$ group homomorphism

$$\begin{array}{ccc} G & \xrightarrow{\rho} & O(V) \\ \Big\uparrow{\scriptstyle \phi_1} & & \Big\downarrow{\scriptstyle \phi_{2*}} \\ \tau^{-1}(O(\phi_2(V))) & \xrightarrow{\tau} & O(W) \end{array}$$

The Category of D_0-Spaces

A D_0-space is a continuous functor $X : D_0 \longrightarrow T$ to the category of compactly generated weak Hausdorff spaces. A morphism of D_0-spaces is a natural transformation.

Model Structure on $D_0 T$

I formulate several model structures on $D_0 T$.

Theorem (Level Model Structure)

The category of D_0-spaces is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper and left proper.

D_0 is a generalized Reedy category in the sense of [Berger and Moerdijk]. And we can formulate a Reedy model structure on $D_0 T$.

Theorem (Reedy Model Structure)

The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category of D_0-spaces.

I'm formulating a global model structure on $D_0 T$ Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

Model Structure on $D_0 T$

I formulate several model structures on $D_0 T$.

Theorem (Level Model Structure)

The category of D_0-spaces is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper and left proper.

D_0 is a generalized Reedy category in the sense of [Berger and Moerdijk]. And we can formulate a Reedy model structure on $D_0 T$.

Theorem (Reedy Model Structure)

The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category of D_0-spaces.

I'm formulating a global model structure on $D_0 T$ Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

Model Structure on $D_0 T$

I formulate several model structures on $D_0 T$.

Theorem (Level Model Structure)

The category of D_0-spaces is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper and left proper.

D_0 is a generalized Reedy category in the sense of [Berger and Moerdijk]. And we can formulate a Reedy model structure on $D_0 T$.

Theorem (Reedy Model Structure)

The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category of D_0-spaces.

I'm formulating a global model structure on $D_0 T$ Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

Model Structure on $D_0 T$

I formulate several model structures on $D_0 T$.

Theorem (Level Model Structure)

The category of D_0-spaces is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper and left proper.

D_0 is a generalized Reedy category in the sense of [Berger and Moerdijk]. And we can formulate a Reedy model structure on $D_0 T$.

Theorem (Reedy Model Structure)

The Reedy cofibrations, Reedy weak equivalences and Reedy fibrations form a model structure, the Reedy model structure, on the category of D_0-spaces.

I'm formulating a global model structure on $D_0 T$ Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

QEll can fit into this global homotopy theory

What is important

Combining the orthogonal $G-$spectra $\{E(G, -)\}$ of quasi-elliptic cohomology together, we get a well-defined D_0-spectra and D_0-FSP. Thus, we can define global quasi-elliptic cohomology in the category of D_0-spectra.

Thank you.

Some references

http://www.math.uiuc.edu/~huan2/Zhen-AMS-2016-Slides.pdf

- Ando, “Isogenies of formal group laws and power operations in the cohomology theories E_n”, Duke J., 1995
- Ando, Hopkins, Strickland, “The sigma orientation is an H_∞ map”, Amer. J. 2004; arXiv:math/0204053
- Adem, Leida, Ruan, "Orbifolds and stringy topology", Cambridge Tracts in Mathematics, 171. 2007.
- Atiyah, "K-theory", second edition, 1989. Notes by D.W.Anderson.
- Atiyah, "Power operations in K-theory", Quart. J. Math. Oxford Ser. (2) 17 1966.
- Atiyah,"Equivariant K-theory and completion", J. Differential Geometry 3 1969.
- Berger, Moerdijk, "On an extension of the notion of Reedy category", Mathematische Zeitschrift, December 2011.
- Bröcker, Dieck, "Representation of Compact Lie Groups", Springer GTM **98** 1985.
- Bruner, May, McClure, Steinberger, "H_∞ ring spectra and their applications", volume 1176 of Lecture Notes in Mathematics, 1986.
- Ganter, “Orbifold genera, product formulas, and power operations”, Adv. Math, 2006; arXiv:math/0407021.
- Ganter, “Stringy power operations in Tate K-theory”, Homology, Homotopy, Appl., 2013; arXiv:math/0701565
- Ganter, “Power operations in orbifold Tate K-theory”; arXiv:1301.2754
- Hirschhorn, "Model categories and their localizations", Mathematical Surveys and Monographs, 99. American Mathematical Society, 2003.
- Hopkins, Kuhn, Ravenel, "Generalized group characters and complex oriented cohomology theories", J. Am. Math. Soc. 13 (2000).
- James, Kerber, "Representation theory of the symmetric group", Encyclopedia of mathematics and its applications v. 16, 1981.
- Katz, Mazur, "Arithmetic moduli of elliptic curves", Annals of Mathematics Studies, vol. 108, 1985.
- Mandell, May, Schwede, Shipley, "Model categories of diagram spectra", Proc. London Math. Soc. 82(2001).
- May, "Equivariant homotopy and cohomology theory", CBMS Regional Conference Series in Mathematics, vol. 91, 1996.
- Rezk, "Quasi-elliptic cohomology", 2011.
- Schommer-Pries, "Central extensions of smooth 2-groups and a finite-dimensional string 2-group", Geom. Topol. 15 (2011).
- Schwede, "Global Homotopy Theory", global.pdf.
- Strickland, "Morava E-theory of symmetric groups", Topology 37 (1998), no. 4.