

# Quasi-elliptic cohomology

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March 29, 2018

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 $\longleftarrow$

even periodic, multiplicative

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- Elliptic curves: elliptic cohomology?

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## Elliptic cohomology

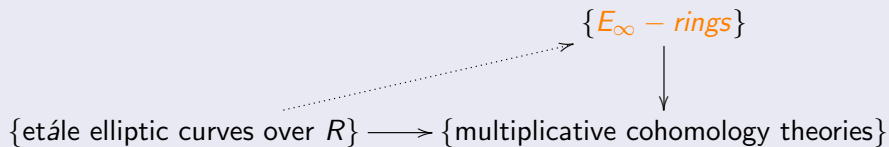
[AHS][Lurie]

$R$ : commutative ring;  $C/R$ : elliptic curve over  $R$ .

$E$  is an elliptic cohomology theory if  $E^0(\text{pt}) \cong R$  and  $\text{Spf}E^0(\mathbb{C}P^\infty) \cong \widehat{C}$ .

## Goerss-Hopkins-Miller-Lurie Theorem

[Lurie]



## Tate K-theory

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**Tate curve:** classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

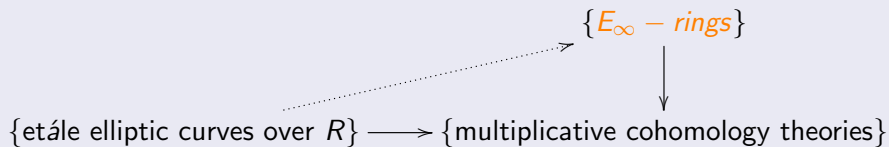
**Tate K-theory:** generalized elliptic cohomology associated to the Tate curve.

### Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

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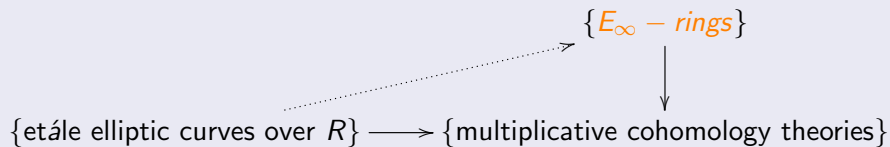
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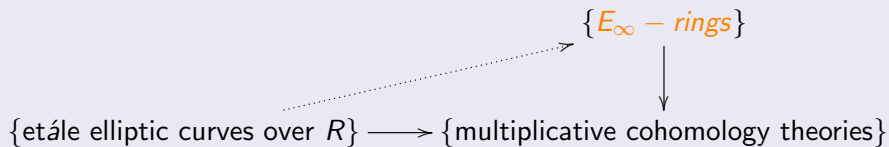
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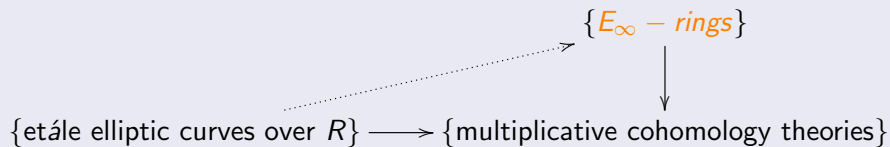
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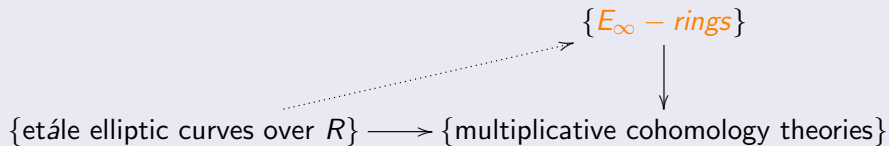
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## Relation with Loop spaces

An old idea by Witten

[Landweber]

$$LX = \mathbb{C}^\infty(S^1, X),$$

$$Ell^*(X) \overset{?}{\longleftrightarrow} K_{\mathbb{T}}^*(LX)$$

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[Ganter]

2007,  $G$ -equivariant Tate K-theory for finite groups  $G$  is modelled on the loop space of a global quotient orbifold.

# Important problems of elliptic cohomology

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Question

Can this construction be generalized to elliptic cohomology?

## Classification Theorems on elliptic curves

### Morava E-theories

- 1995, Matthew Ando: a classification of the level- $p^k$  structure of its formal group.
- 1998, Neil Strickland: the Morava  $E$ -theory of the symmetric group  $\Sigma_n$  modulo a certain transfer ideal classifies the power subgroups of rank  $n$  of its formal group.
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Are the geometric structures on elliptic curves classified in the same way?

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# Important problems of elliptic cohomology

## Equivariant elliptic cohomology

Ginzburg, Kapranov and Vasserot's Conjecture (1995) [GRV]

Any elliptic curve  $A$  gives rise to a unique equivariant elliptic cohomology theory, natural in  $A$ .

Relevant Work [Gepner]

1999, David Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms.

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Can we construct equivariant spectra for them?

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## Relation with Tate K-theory

$$QEII_G^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K_{Tate}^*(X // G).$$

- Contain all the information of Tate K-theory
- $QEII_G^*(X)$  is a  $\mathbb{Z}[q^{\pm}]$ -module.

 $QEII$  as equivariant  $K$ -theories

$$QEII_G(X) := K_{orb}(\Lambda(X // G)) \cong \prod_{g \in G_{conj}^{tors}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

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## Representation theory

- Restriction map:  $RG \longrightarrow RH$ ;

## Equivariant K-theory

- Restriction map:  $K_G(X) \longrightarrow K_H(X)$ ;

## Quasi-elliptic cohomology

- Restriction map:  $QEll_G(X) \longrightarrow QEll_H(X)$ ;

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- Restriction map:  $RG \longrightarrow RH$ ;
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- $RG \otimes RH \longrightarrow R(G \times H)$ .

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- **Change-of-group isomorphism:  $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$ ;**

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- $K_G^*(-)$  can be represented by an orthogonal  $G$ -spectrum;
- **Global K-theory exists.**

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## What is "Loop"?

a morphism in some category from  $S^1$  to  $X//G$ .

## What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from  $S^1//*$  to  $X//G$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

- $\pi$  : principal  $G$ -bundle over  $S^1$
- $f$  :  $G$ -equivariant;
- Morphism  $\mathcal{P} \rightarrow \mathcal{P}'$ :  $G$ -bundle map  $\alpha : P \rightarrow P'$

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## What is "Loop"?

a morphism in some category from  $S^1$  to  $X//G$ .

## What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from  $S^1//_*$  to  $X//G$ 

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

- $\pi$  : principal  $G$ -bundle over  $S^1$
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Where is the rotation? **Add it to the loop space!**

$Loop^{ext}(X//G)$

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$\Lambda(X//G)$ : a subgroupoid of  $Loop^{ext}(X//G)$  consisting of constant loops.

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The image of  $f$  contained in a single  $G$ -orbit.

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- When  $G$  is finite,  $\Lambda(X // G) = GhLoop(X // G)$ ;
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# Power operation of equivariant cohomology theories

## Power Operation of K-theory

[Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

## Power Operation of equivariant K-theory

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$$P_n : K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

## Wreath product $G \wr \Sigma_n$

$$(g_1, \dots, g_n, \sigma) \cdot (h_1, \dots, h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \dots, g_n h_{\sigma^{-1}(n)}, \sigma \tau).$$

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## Definition of Equivariant Power Operation

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## Atiyah's Power Operation

[Ganter]

$V$ : a vector bundle over  $\Lambda(X//G)$ .

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}^n}$  defines an operation

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## The Elliptic Power Operation

[Huan]

$$\mathbb{P}_n = \prod_{(\underline{g}, \sigma) \in (G|\Sigma_n)_{conj}^{tors}} \mathbb{P}_{(\underline{g}, \sigma)}:$$

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## Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

## Example ( $G = e$ )

$QEll_G^*(X) = K_T^*(X)$ . For each  $\sigma \in \Sigma_n$ ,  $\mathbb{P}_{(\underline{1}, \sigma)}(x) = \boxtimes_k \boxtimes_{(i_1, \dots, i_k)} (x)_k$ .

When  $n = 2$ ,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1}, (1)(1))}(x), \mathbb{P}_{(\underline{1}, (12))}(x)) = (x \boxtimes x, (x)_2).$$

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## Theorem (Huan)

The Tate  $K$ -theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{\text{Tate}}(\text{pt} // \Sigma_N) / I_{\text{tr}}^{\text{Tate}} \cong \prod_{N=de} \mathbb{Z}((q)) [q'_s{}^{\pm}] / \langle q^d - q'_s{}^e \rangle,$$

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### Motivating example: Quillen model structure on topological spaces

Represent the standard homotopy theory of CW-complexes.

- weak homotopy equivalence
- Serre fibration
- retract of relative cell complex

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**Definition:** A model structure on a category  $\mathcal{C}$

Three distinguished classes of morphisms:

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## Morphisms

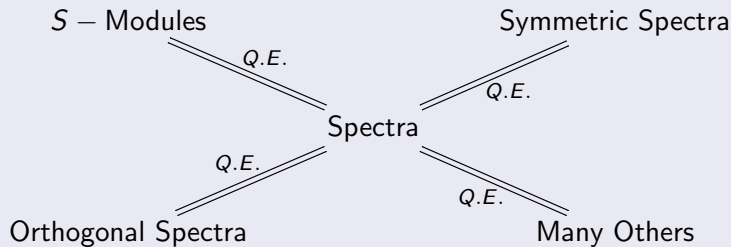
Quillen adjunction:  $(L \dashv R) : \mathcal{C} \begin{matrix} \xleftarrow{R} \\ \xrightarrow{L} \end{matrix} \mathcal{D}$ .

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# Why do we care equivariant orthogonal spectra?

Right Philosophy for Stable Homotopy Theory (1990s)

[MMSS]





# Why do we care equivariant orthogonal spectra?

Equivariant Stable Homotopy Theory (2000s)

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$S_G$  – Modules

Orthogonal  $G$ -Spectra

*Q.E.*

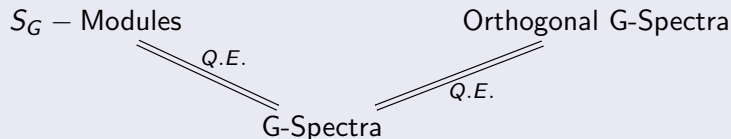
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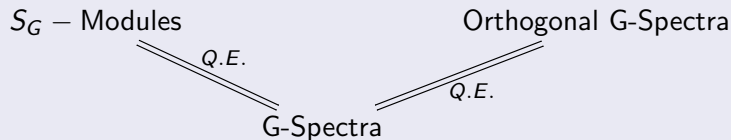
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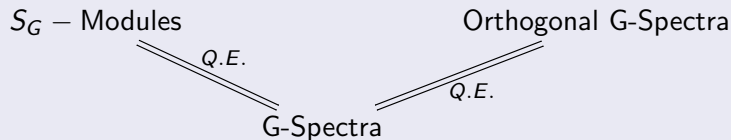
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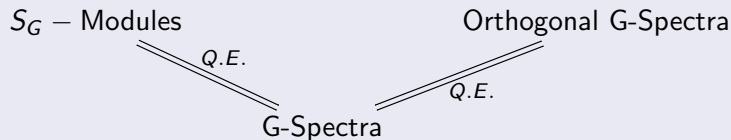
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$Top_G$ : the category of based  $G$ -spaces and continuous based maps.

$\mathcal{I}_G$ -space

A  $G$ -continuous functor  $X : \mathcal{I}_G \rightarrow Top_G$ .

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An  $\mathcal{I}_G$ -space  $X$  with a natural transformation  $X(-) \wedge S^- \rightarrow X(- \oplus -)$  such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An  $\mathcal{I}_G$ -FSP is an  $\mathcal{I}_G$ -space  $X$  with

- a unit  $\eta : S \rightarrow X$ ;
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⇒ **global homotopy theory**

**Prominent examples:** equivariant stable homotopy, equivariant K-theory, equivariant bordism.

The category  $\mathbb{L}$

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$GwS$ : the new category of orthogonal  $G$ -spectra

[Huan]

$GwS :=$  the homotopy category of the category of orthogonal  $G$ -spectra with the weak equivalence defined by

$$X \sim Y \text{ if } \pi_0(X(V)) = \pi_0(Y(V)),$$

for each faithful  $G$ -representation  $V$ .

An orthogonal  $G$ -spectrum  $X$  in  $GwS$  is said to represent a theory  $H_G^*$  if we have a natural map

$$\pi_0(X(V)) = H_G^V(S^0),$$

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## Theorem

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There exists a commutative  $\mathcal{I}_G$ -FSP  $(E(G, -), \eta, \mu)$  representing  $QEII_G^*(-)$  in  $GwS$ .

The construction can be generalized.

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There is a well-defined functor from the full subcategory consisting of  $\mathcal{I}_G$ -FSP in  $GwS$  to itself.

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- $u: X \mapsto X\langle G \rangle$  underlying orthogonal  $G$ -spectrum;
- arise: yes iff for any trivial  $G$ -representation  $V$ , the  $G$ -action on  $Y(V)$  is trivial.

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No.

For a trivial  $G$ -representation  $V$ , the  $G$ -action on  $E(G, V)$  is not trivial.

## Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category  $D_0$ : add restriction maps to  $\mathbb{L}$

This is a generalized Reedy category.

- Objects:  $(G, V)$ ;
- Morphisms:
  - linear isometric embedding: raising degree;
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**Chenchang Zhu** suggests, other than Gepner's and my approach, we can use the geometric object **two-vector bundles** representing elliptic cohomology to establish equivariant elliptic cohomology theories.

## Global homotopy theory for elliptic cohomology

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Quasi-elliptic cohomology serves as a breakthrough point of the construction.

- Continue with the global homotopy theory under construction: construct the global model structure and the corresponding stable global homotopy theory.
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**Ganter's conjecture:** the HKR character theory for orbifold Tate K-theory to be established via **Stapleton's** framework of **transchromatic character theory**.

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I hope so.

*Thank you.*

<http://gagp.sysu.edu.cn/zhenhuan/Zhen-CAS-2018-Slides.pdf>

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