Quasi-elliptic cohomology

Zhen Huan

Sun Yat-sen University

March 29, 2018

$$\begin{tikzcd}\text{generalized cohomology theories} \arrow[r, bend left, "\text{1st Chern class}"] & \text{formal groups} \arrow[l, bend left]\end{tikzcd}$$

even periodic, multiplicative

Elliptic cohomology theory

$$\text{generalized cohomology theories} \underset{}{\overset{\text{1st Chern class}}{\rightleftarrows}} \text{formal groups}$$

even periodic, multiplicative

commutative 1-dimensional formal groups

- The additive formal group $\mathbb{G}_a$: periodic Eilenberg-MacLane spectrum.

Elliptic cohomology theory

$$\text{generalized cohomology theories} \underset{}{\overset{\text{1st Chern class}}{\rightleftarrows}} \text{formal groups}$$

even periodic, multiplicative

commutative 1-dimensional formal groups

- The additive formal group $\mathbb{G}_a$: periodic Eilenberg-MacLane spectrum.
- The multiplicative formal group $\mathbb{G}_m$: complex K-theory.

$$\text{generalized cohomology theories} \underset{}{\overset{\text{1st Chern class}}{\rightleftarrows}} \text{formal groups}$$

even periodic, multiplicative

commutative 1-dimensional formal groups

- The additive formal group $\mathbb{G}_a$: periodic Eilenberg-MacLane spectrum.
- The multiplicative formal group $\mathbb{G}_m$: complex K-theory.
- Elliptic curves: elliptic cohomology?

Elliptic cohomology theory

$$\text{generalized cohomology theories} \underset{}{\overset{\text{1st Chern class}}{\rightleftarrows}} \text{formal groups}$$

even periodic, multiplicative

commutative 1-dimensional formal groups

- The additive formal group $\mathbb{G}_a$: periodic Eilenberg-MacLane spectrum.
- The multiplicative formal group $\mathbb{G}_m$: complex K-theory.
- Elliptic curves: elliptic cohomology?

Elliptic cohomology [AHS][Lurie]

R: commutative ring; C/R: elliptic curve over R.
E is an elliptic cohomology theory if $E^0(\mathrm{pt}) \cong R$ and $\mathrm{Spf} E^0(\mathbb{C}P^\infty) \cong \widehat{C}$.

Elliptic cohomology

Goerss-Hopkins-Miller-Lurie Theorem [Lurie]

$$\begin{tikzcd} & \{E_\infty - rings\} \arrow[d] \\ \{\text{etále elliptic curves over } R\} \arrow[r] \arrow[ur, dotted] & \{\text{multiplicative cohomology theories}\} \end{tikzcd}$$

Tate K-theory [AHS]

Tate curve: classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

Elliptic cohomology

Goerss-Hopkins-Miller-Lurie Theorem [Lurie]

$$
\begin{tikzcd}
 & \{{\color{orange}E_\infty - rings}\} \arrow[d] \\
\{\text{etále elliptic curves over } R\} \arrow[ur, dotted] \arrow[r] & \{\text{multiplicative cohomology theories}\}
\end{tikzcd}
$$

Tate K-theory [AHS]

Tate curve: classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

Elliptic cohomology

Goerss-Hopkins-Miller-Lurie Theorem [Lurie]

$$\begin{tikzcd}
 & \{E_\infty - rings\} \arrow[d] \\
\{\text{etále elliptic curves over } R\} \arrow[r] \arrow[ur, dashed] & \{\text{multiplicative cohomology theories}\}
\end{tikzcd}$$

Tate K-theory [AHS]

Tate curve: classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

Goerss-Hopkins-Miller-Lurie Theorem [Lurie]

$$\begin{tikzcd}
& \{{\color{orange}E_\infty - rings}\} \arrow[d] \\
\{\text{etále elliptic curves over } R\} \arrow[ur, dotted] \arrow[r] & \{\text{multiplicative cohomology theories}\}
\end{tikzcd}$$

Tate K-theory [AHS]

Tate curve: classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

Elliptic cohomology

Goerss-Hopkins-Miller-Lurie Theorem [Lurie]

$$\begin{tikzcd} & \{E_\infty - rings\} \arrow[d] \\ \{\text{etále elliptic curves over } R\} \arrow[r] \arrow[ur, dotted] & \{\text{multiplicative cohomology theories}\} \end{tikzcd}$$

Tate K-theory [AHS]

Tate curve: classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.
Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

Elliptic cohomology

Goerss-Hopkins-Miller-Lurie Theorem [Lurie]

$$\begin{tikzcd} & \{E_\infty - rings\} \arrow[d] \\ \{\text{et}\acute{a}\text{le elliptic curves over } R\} \arrow[r] \arrow[ur, dotted] & \{\text{multiplicative cohomology theories}\} \end{tikzcd}$$

Tate K-theory [AHS]

Tate curve: classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

Important problems of elliptic cohomology

Relation with Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^{\infty}(S^1, X),$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

Important problems of elliptic cohomology

Relation with Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^{\infty}(S^1, X),$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

Important problems of elliptic cohomology

Relation with Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^{\infty}(S^1, X),$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

It's **SURPRISINGLY** difficult to make this idea precise.

Important problems of elliptic cohomology

Relation with Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^\infty(S^1, X),\ S^1 \circlearrowright \mathbb{T} \qquad X \circlearrowright G$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

It's **SURPRISINGLY** difficult to make this idea precise.

Important problems of elliptic cohomology

Relation with Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^\infty(S^1, X),\ S^1 \circlearrowright \mathbb{T} \qquad X \circlearrowright G$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

It's **SURPRISINGLY** difficult to make this idea precise.

Relevant Work [Ganter]

2007, $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold.

Important problems of elliptic cohomology

Relation with Loop spaces

An old idea by Witten [Landweber]

$$LX = \mathbb{C}^\infty(S^1, X),\ S^1 \circlearrowleft \mathbb{T} \quad X \circlearrowleft G$$

$$Ell^*(X) \overset{?}{\leftrightsquigarrow} K^*_{\mathbb{T}}(LX)$$

It's **SURPRISINGLY** difficult to make this idea precise.

Relevant Work [Ganter]

2007, $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold.

Question

Can this construction be generalized to elliptic cohomology?

Important problems of elliptic cohomology

Classification Theorems on elliptic curves

Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: Same result for generalized Morava E-theories. Via transchromatic character theory.

$$\text{the homotopy theory} \xleftrightarrow{\text{Power Operation}} \text{its formal group.}$$

Question

Are the geometric structures on elliptic curves classified in the same way?

Important problems of elliptic cohomology

Classification Theorems on elliptic curves

Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: Same result for generalized Morava E-theories. Via transchromatic character theory.

$$\text{the homotopy theory} \xleftrightarrow{\text{Power Operation}} \text{its formal group.}$$

Question

Are the geometric structures on elliptic curves classified in the same way?

Important problems of elliptic cohomology

Classification Theorems on elliptic curves

Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: Same result for generalized Morava E-theories. Via transchromatic character theory.

$$\text{the homotopy theory} \xleftrightarrow{\text{Power Operation}} \text{its formal group.}$$

Question

Are the geometric structures on elliptic curves classified in the same way?

Important problems of elliptic cohomology

Classification Theorems on elliptic curves

Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: Same result for generalized Morava E-theories. Via transchromatic character theory.

$$\text{the homotopy theory} \xleftrightarrow{\text{Power Operation}} \text{its formal group.}$$

Question

Are the geometric structures on elliptic curves classified in the same way?

Important problems of elliptic cohomology

Classification Theorems on elliptic curves

Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: Same result for generalized Morava E-theories. Via transchromatic character theory.

$$\text{the homotopy theory} \xleftrightarrow{\text{Power Operation}} \text{its formal group.}$$

Question

Are the geometric structures on elliptic curves classified in the same way?

Important problems of elliptic cohomology

Classification Theorems on elliptic curves

Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: Same result for generalized Morava E-theories. Via transchromatic character theory.

$$\text{the homotopy theory} \xleftrightarrow{\text{Power Operation}} \text{its formal group.}$$

Question

Are the geometric structures on elliptic curves classified in the same way?

Important problems of elliptic cohomology

Classification Theorems on elliptic curves

Morava E-theories

- 1995, Matthew Ando: a classification of the level$-p^k$ structure of its formal group.
- 1998, Neil Strickland: the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: Same result for generalized Morava E-theories. Via transchromatic character theory.

$$\text{the homotopy theory} \xleftrightarrow{\text{Power Operation}} \text{its formal group.}$$

Question

Are the geometric structures on elliptic curves classified in the same way?

Important problems of elliptic cohomology

Equivariant elliptic cohomology

Ginzburg, Kapranov and Vasserot's Conjecture (1995) [GRV]

Any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A.

Relevant Work [Gepner]

1999, David Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms.

Question

Can we construct equivariant spectra for them?

Further Question

How can we construct global elliptic cohomology?

Important problems of elliptic cohomology

Equivariant elliptic cohomology

Ginzburg, Kapranov and Vasserot's Conjecture (1995) [GRV]

Any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A.

Relevant Work [Gepner]

1999, David Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms.

Question

Can we construct equivariant spectra for them?

Further Question

How can we construct global elliptic cohomology?

Important problems of elliptic cohomology

Equivariant elliptic cohomology

Ginzburg, Kapranov and Vasserot's Conjecture (1995) [GRV]

Any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A.

Relevant Work [Gepner]

1999, David Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms.

Question

Can we construct equivariant spectra for them?

Further Question

How can we construct global elliptic cohomology?

Important problems of elliptic cohomology

Equivariant elliptic cohomology

Ginzburg, Kapranov and Vasserot's Conjecture (1995) [GRV]

Any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A.

Relevant Work [Gepner]

1999, David Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms.

Question

Can we construct equivariant spectra for them?

Further Question

How can we construct global elliptic cohomology?

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong K^*_{Tate}(X/\!\!/G).$$

- Contain all the information of Tate K-theory.
- $QEll^*_G(X)$ is a $\mathbb{Z}[q^\pm]-$module.

$QEll$ as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X/\!\!/G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

- Contain all the information of Tate K-theory.
- $QEll^*_G(X)$ is a $\mathbb{Z}[q^{\pm}]-$module.

$QEll$ as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

- Contain all the information of Tate K-theory.
- $QEll^*_G(X)$ is a $\mathbb{Z}[q^{\pm}]-$module.

$QEll$ as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

- Contain all the information of Tate K-theory.
- $QEll^*_G(X)$ is a $\mathbb{Z}[q^{\pm}]-$module.

$QEll$ as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Quasi-elliptic cohomology [Ganter][Rezk][Huan]

Relation with Tate K-theory

$$QEll_G^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

- Contain all the information of Tate K-theory.
- $QEll_G^*(X)$ is a $\mathbb{Z}[q^{\pm}]-$module.

QEll as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

- Contain all the information of Tate K-theory.
- $QEll^*_G(X)$ is a $\mathbb{Z}[q^{\pm}]-$module.

$QEll$ as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Quasi-elliptic cohomology [Ganter][Rezk][Huan]

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!\!/ G).$$

- Contain all the information of Tate K-theory.
- $QEll^*_G(X)$ is a $\mathbb{Z}[q^\pm]-$module.

$QEll$ as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Relation with Tate K-theory

$$QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K^*_{Tate}(X /\!/ G).$$

- Contain all the information of Tate K-theory.
- $QEll^*_G(X)$ is a $\mathbb{Z}[q^{\pm}]-$module.

$QEll$ as equivariant $K-$theories

$$QEll_G(X) := K_{orb}(\Lambda(X /\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- Equivariant K-theory has been fully studied.
- Reduce questions into representation theory.
- Neat construction.

Properties of Quasi-elliptic cohomology

Representation theory

- Restriction map: $RG \longrightarrow RH$;

Equivariant K-theory

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;

Quasi-elliptic cohomology

- Restriction map: $QEll_G(X) \longrightarrow QEll_H(X)$;

Properties of Quasi-elliptic cohomology

Representation theory

- Restriction map: $RG \longrightarrow RH$;
- Induced map: $RH \longrightarrow RG$.

Equivariant K-theory

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;

Quasi-elliptic cohomology

- Restriction map: $QEll_G(X) \longrightarrow QEll_H(X)$;
- Induced map: $QEll_H(X) \longrightarrow QEll_G(X)$;

Properties of Quasi-elliptic cohomology

Representation theory

- Restriction map: $RG \longrightarrow RH$;
- Induced map: $RH \longrightarrow RG$.
- $RG \otimes RH \longrightarrow R(G \times H)$.

Equivariant K-theory

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Künneth map: $\mathrm{K}_G^*(X) \otimes K_H^*(Y) \longrightarrow K_{G\times H}^*(X \times Y)$;

Quasi-elliptic cohomology

- Restriction map: $QEll_G(X) \longrightarrow QEll_H(X)$;
- Induced map: $QEll_H(X) \longrightarrow QEll_G(X)$;
- Künneth map: $\mathrm{QEll}_G^*(X) \widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} QEll_H^*(Y) \longrightarrow QEll_{G\times H}^*(X \times Y)$.

Properties of Quasi-elliptic cohomology

Equivariant K-theory

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Künneth map: $\mathrm{K}^*_G(X) \otimes K^*_H(Y) \longrightarrow K^*_{G\times H}(X \times Y)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

Quasi-elliptic cohomology

- Restriction map: $QEll_G(X) \longrightarrow QEll_H(X)$;
- Induced map: $QEll_H(X) \longrightarrow QEll_G(X)$;
- Künneth map: $\mathrm{QEll}^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y)$.
- Change-of-group isomorphism: $QEll_G(Y \times_H G) \xrightarrow{\cong} QEll_H(Y)$;

Properties of Quasi-elliptic cohomology

Equivariant K-theory

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Künneth map: $\mathrm{K}^*_G(X) \otimes K^*_H(Y) \longrightarrow K^*_{G\times H}(X \times Y)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;
- $K^*_G(-)$ can be represented by an orthogonal $G-$spectrum;

Quasi-elliptic cohomology

- Restriction map: $QEll_G(X) \longrightarrow QEll_H(X)$;
- Induced map: $QEll_H(X) \longrightarrow QEll_G(X)$;
- Künneth map: $\mathrm{QEll}^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} QEll^*_H(Y) \longrightarrow QEll^*_{G\times H}(X \times Y)$.
- Change-of-group isomorphism: $QEll_G(Y \times_H G) \xrightarrow{\cong} QEll_H(Y)$;

Properties of Quasi-elliptic cohomology

Equivariant K-theory

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Künneth map: $\mathrm{K}_G^*(X) \otimes K_H^*(Y) \longrightarrow K_{G\times H}^*(X \times Y)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;
- $K_G^*(-)$ can be represented by an orthogonal $G-$spectrum;
- Global K-theory exists.

Quasi-elliptic cohomology

- Restriction map: $QEll_G(X) \longrightarrow QEll_H(X)$;
- Induced map: $QEll_H(X) \longrightarrow QEll_G(X)$;
- Künneth map: $\mathrm{QEll}_G^*(X) \widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} QEll_H^*(Y) \longrightarrow QEll_{G\times H}^*(X \times Y)$.
- Change-of-group isomorphism: $QEll_G(Y \times_H G) \xrightarrow{\cong} QEll_H(Y)$;

What is "Loop"?

a morphism in some category from S^1 to $X/\!/G$.

What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from $S^1/\!/*$ to $X/\!/G$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{tikzcd} S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\ & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

What is "Loop"?

a morphism in some category from S^1 to $X /\!/ G$.

What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from $S^1/\!/*$ to $X/\!/G$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;
- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{tikzcd} S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\ & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

What is "Loop"?

a morphism in some category from S^1 to $X /\!\!/ G$.

What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from $S^1 /\!\!/ *$ to $X /\!\!/ G$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{tikzcd} S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\ & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

What is "Loop"?

a morphism in some category from S^1 to $X /\!\!/ G$.

What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from $S^1 /\!\!/ *$ to $X /\!\!/ G$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{tikzcd} S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\ & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

What is "Loop"?

a morphism in some category from S^1 to $X /\!\!/ G$.

What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from $S^1 /\!\!/ *$ to $X /\!\!/ G$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{tikzcd} S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\ & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

What is "Loop"?

a morphism in some category from S^1 to $X/\!\!/G$.

What is the category?

The localization of Lie groupoids w.r.t. equivalence of Lie groupoids.

The Loop space := Bibundles from $S^1/\!\!/*$ to $X/\!\!/G$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;

- Morphism $\mathcal{P} \longrightarrow \mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{tikzcd} S^1 & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\ & P' \arrow[ul, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

Where is the rotation? Add it to the loop space!

$Loop^{ext}(X /\!\!/ G)$

- the same objects as $Loop_1(X /\!\!/ G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{tikzcd} S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"'] & X \\ S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

The groupoid we really need

$\Lambda(X /\!\!/ G)$: a subgroupoid of $Loop^{ext}(X /\!\!/ G)$ consisting of constant loops.

$$QEll_G(X) = K_{orb}(\Lambda(X /\!\!/ G))$$

Where is the rotation? Add it to the loop space!

$Loop^{ext}(X /\!\!/ G)$

- the same objects as $Loop_1(X /\!\!/ G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{tikzcd} S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"'] & X \\ S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

The groupoid we really need

$\Lambda(X /\!\!/ G)$: a subgroupoid of $Loop^{ext}(X /\!\!/ G)$ consisting of constant loops.

$$QEll_G(X) = K_{orb}(\Lambda(X /\!\!/ G))$$

Loop Space Construction [Rezk][Huan]

Where is the rotation? Add it to the loop space!

$Loop^{ext}(X/\!\!/G)$

- the same objects as $Loop_1(X/\!\!/G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{tikzcd}
S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"] & X \\
S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

The groupoid we really need

$\Lambda(X/\!\!/G)$: a subgroupoid of $Loop^{ext}(X/\!\!/G)$ consisting of constant loops.

$$QEll_G(X) = K_{orb}(\Lambda(X/\!\!/G))$$

Where is the rotation? Add it to the loop space!

$Loop^{ext}(X /\!\!/ G)$

- the same objects as $Loop_1(X /\!\!/ G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{tikzcd} S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"'] & X \\ S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

The groupoid we really need

$\Lambda(X /\!\!/ G)$: a subgroupoid of $Loop^{ext}(X /\!\!/ G)$ consisting of constant loops.

$$QEll_G(X) = K_{orb}(\Lambda(X /\!\!/ G))$$

Where is the rotation? Add it to the loop space!

$Loop^{ext}(X /\!/ G)$

- the same objects as $Loop_1(X /\!/ G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{tikzcd}
S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"'] & X \\
S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

The groupoid we really need

$\Lambda(X /\!/ G)$: a subgroupoid of $Loop^{ext}(X /\!/ G)$ consisting of constant loops.

$$QEll_G(X) = K_{orb}(\Lambda(X /\!/ G))$$

Where is the rotation? Add it to the loop space!

$Loop^{ext}(X /\!/ G)$

- the same objects as $Loop_1(X /\!/ G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{tikzcd}
S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"'] & X \\
S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] &
\end{tikzcd}$$

The groupoid we really need

$\Lambda(X /\!/ G)$: a subgroupoid of $Loop^{ext}(X /\!/ G)$ consisting of constant loops.

$$QEll_G(X) = K_{orb}(\Lambda(X /\!/ G))$$

Where is the rotation? Add it to the loop space!

$Loop^{ext}(X /\!\!/ G)$

- the same objects as $Loop_1(X /\!\!/ G)$;
- $(t, \alpha) : \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\} \longrightarrow \{S^1 \xleftarrow{\pi'} P' \xrightarrow{f'} X\}$
  - $t \in \mathbb{T}$
  - $\alpha : P \longrightarrow P'$: $G-$bundle map

$$\begin{tikzcd} S^1 \arrow[d, "t"'] & P \arrow[l, "\pi"'] \arrow[r, "f"] \arrow[d, "\alpha"'] & X \\ S^1 & P' \arrow[l, "\pi'"] \arrow[ur, "f'"'] & \end{tikzcd}$$

The groupoid we really need

$\Lambda(X /\!\!/ G)$: a subgroupoid of $Loop^{ext}(X /\!\!/ G)$ consisting of constant loops.

$$QEll_G(X) = K_{orb}(\Lambda(X /\!\!/ G))$$

A Better Model: Ghost Loops [Rezk]

$GhLoop(X/\!\!/G)$:= the full subgroupoid of $Loop^{ext}(X/\!\!/G)$ consisting of

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

The image of f contained in a single $G-$orbit.

$$\Lambda(X/\!\!/G) \subseteq GhLoop(X/\!\!/G) \subseteq Loop^{ext}(X/\!\!/G)$$

Good features of $GhLoop(X/\!\!/G)$

- When G is finite, $\Lambda(X/\!\!/G) = GhLoop(X/\!\!/G)$;
- Mayer-Vietoris Property:
  $GhLoop(X/\!\!/G) \cong GhLoop(U/\!\!/G) \cup_{GhLoop((U\cap V)/\!\!/G)} GhLoop(V/\!\!/G)$.
- Change-of-Group Property: $GhLoop((G/H)/\!\!/G) \cong GhLoop(\mathrm{pt}/\!\!/H)$.

$H^*(-)$: equivariant cohomology theory,

$$X/\!\!/G \mapsto H^*(GhLoop(X/\!\!/G))$$

gives a new equivariant cohomology theory.

A Better Model: Ghost Loops [Rezk]

$GhLoop(X /\!\!/ G)$:= the full subgroupoid of $Loop^{ext}(X /\!\!/ G)$ consisting of

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

The image of f contained in a single $G-$orbit.

$$\Lambda(X /\!\!/ G) \subseteq GhLoop(X /\!\!/ G) \subseteq Loop^{ext}(X /\!\!/ G)$$

Good features of $GhLoop(X /\!\!/ G)$

- When G is finite, $\Lambda(X /\!\!/ G) = GhLoop(X /\!\!/ G)$;
- Mayer-Vietoris Property:
  $GhLoop(X /\!\!/ G) \cong GhLoop(U /\!\!/ G) \cup_{GhLoop((U \cap V) /\!\!/ G)} GhLoop(V /\!\!/ G)$.
- Change-of-Group Property: $GhLoop((G/H) /\!\!/ G) \cong GhLoop(\mathrm{pt} /\!\!/ H)$.

$H^*(-)$: equivariant cohomology theory,

$$X /\!\!/ G \mapsto H^*(GhLoop(X /\!\!/ G))$$

gives a new equivariant cohomology theory.

A Better Model: Ghost Loops [Rezk]

$GhLoop(X /\!\!/ G)$:= the full subgroupoid of $Loop^{ext}(X /\!\!/ G)$ consisting of

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

The image of f contained in a single $G-$orbit.

$$\Lambda(X /\!\!/ G) \subseteq GhLoop(X /\!\!/ G) \subseteq Loop^{ext}(X /\!\!/ G)$$

Good features of $GhLoop(X /\!\!/ G)$

- When G is finite, $\Lambda(X /\!\!/ G) = GhLoop(X /\!\!/ G)$;
- Mayer-Vietoris Property: $GhLoop(X /\!\!/ G) \cong GhLoop(U /\!\!/ G) \cup_{GhLoop((U \cap V) /\!\!/ G)} GhLoop(V /\!\!/ G)$.
- Change-of-Group Property: $GhLoop((G/H) /\!\!/ G) \cong GhLoop(\mathrm{pt} /\!\!/ H)$.

$H^*(-)$: equivariant cohomology theory,

$$X /\!\!/ G \mapsto H^*(GhLoop(X /\!\!/ G))$$

gives a new equivariant cohomology theory.

$GhLoop(X/\!\!/G)$:= the full subgroupoid of $Loop^{ext}(X/\!\!/G)$ consisting of

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

The image of f contained in a single $G-$orbit.

$$\Lambda(X/\!\!/G) \subseteq GhLoop(X/\!\!/G) \subseteq Loop^{ext}(X/\!\!/G)$$

Good features of $GhLoop(X/\!\!/G)$

- When G is finite, $\Lambda(X/\!\!/G) = GhLoop(X/\!\!/G)$;
- Mayer-Vietoris Property: $GhLoop(X/\!\!/G) \cong GhLoop(U/\!\!/G) \cup_{GhLoop((U\cap V)/\!\!/G)} GhLoop(V/\!\!/G)$.
- Change-of-Group Property: $GhLoop((G/H)/\!\!/G) \cong GhLoop(\mathrm{pt}/\!\!/H)$.

$H^*(-)$: equivariant cohomology theory,

$$X/\!\!/G \mapsto H^*(GhLoop(X/\!\!/G))$$

gives a new equivariant cohomology theory.

$GhLoop(X /\!\!/ G)$:= the full subgroupoid of $Loop^{ext}(X /\!\!/ G)$ consisting of

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

The image of f contained in a single $G-$orbit.

$$\Lambda(X /\!\!/ G) \subseteq GhLoop(X /\!\!/ G) \subseteq Loop^{ext}(X /\!\!/ G)$$

Good features of $GhLoop(X /\!\!/ G)$

- When G is finite, $\Lambda(X /\!\!/ G) = GhLoop(X /\!\!/ G)$;
- Mayer-Vietoris Property: $GhLoop(X /\!\!/ G) \cong GhLoop(U /\!\!/ G) \cup_{GhLoop((U \cap V) /\!\!/ G)} GhLoop(V /\!\!/ G)$.
- Change-of-Group Property: $GhLoop((G/H) /\!\!/ G) \cong GhLoop(\mathrm{pt} /\!\!/ H)$.

$H^*(-)$: equivariant cohomology theory,

$$X /\!\!/ G \mapsto H^*(GhLoop(X /\!\!/ G))$$

gives a new equivariant cohomology theory.

A Better Model: Ghost Loops [Rezk]

$GhLoop(X/\!\!/G)$:= the full subgroupoid of $Loop^{ext}(X/\!\!/G)$ consisting of

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

The image of f contained in a single $G-$orbit.

$$\Lambda(X/\!\!/G) \subseteq GhLoop(X/\!\!/G) \subseteq Loop^{ext}(X/\!\!/G)$$

Good features of $GhLoop(X/\!\!/G)$

- When G is finite, $\Lambda(X/\!\!/G) = GhLoop(X/\!\!/G)$;
- Mayer-Vietoris Property: $GhLoop(X/\!\!/G) \cong GhLoop(U/\!\!/G) \cup_{GhLoop((U\cap V)/\!\!/G)} GhLoop(V/\!\!/G)$.
- Change-of-Group Property: $GhLoop((G/H)/\!\!/G) \cong GhLoop(\mathrm{pt}/\!\!/H)$.

$H^*(-)$: equivariant cohomology theory,

$$X/\!\!/G \mapsto H^*(GhLoop(X/\!\!/G))$$

gives a new equivariant cohomology theory.

$GhLoop(X/\!\!/G)$:= the full subgroupoid of $Loop^{ext}(X/\!\!/G)$ consisting of

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

The image of f contained in a single $G-$orbit.

$$\Lambda(X/\!\!/G) \subseteq GhLoop(X/\!\!/G) \subseteq Loop^{ext}(X/\!\!/G)$$

Good features of $GhLoop(X/\!\!/G)$

- When G is finite, $\Lambda(X/\!\!/G) = GhLoop(X/\!\!/G)$;
- Mayer-Vietoris Property: $GhLoop(X/\!\!/G) \cong GhLoop(U/\!\!/G) \cup_{GhLoop((U\cap V)/\!\!/G)} GhLoop(V/\!\!/G)$.
- Change-of-Group Property: $GhLoop((G/H)/\!\!/G) \cong GhLoop(\mathrm{pt}/\!\!/H)$.

$H^*(-)$: equivariant cohomology theory,

$$X/\!\!/G \mapsto H^*(GhLoop(X/\!\!/G))$$

gives a new equivariant cohomology theory.

Power Operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G\wr\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau)$.
Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)})$.

Definition of Equivariant Power Operation [May][Ganter]

$$P_n : E_G(X) \longrightarrow E_{G\wr\Sigma_n}(X^{\times n})$$

satisfying some axioms.

Power Operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{????}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau)$.

Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)})$.

Definition of Equivariant Power Operation [May][Ganter]

$$P_n : E_G(X) \longrightarrow E_{G \wr \Sigma_n}(X^{\times n})$$

satisfying some axioms.

Power operation of equivariant cohomology theories

Power Operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G\wr\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau)$.

Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)})$.

Definition of Equivariant Power Operation [May][Ganter]

$$P_n : E_G(X) \longrightarrow E_{G\wr\Sigma_n}(X^{\times n})$$

satisfying some axioms.

Power operation of equivariant cohomology theories

Power Operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G\wr\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau)$.
Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)})$.

Definition of Equivariant Power Operation [May][Ganter]

$$P_n : E_G(X) \longrightarrow E_{G\wr\Sigma_n}(X^{\times n})$$

satisfying some axioms.

Power operation of equivariant cohomology theories

Power Operation of K-theory [Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory [Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$(g_1, \cdots g_n, \sigma) \cdot (h_1, \cdots h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \cdots g_n h_{\sigma^{-1}(n)}, \sigma\tau).$
Group action: $(x_1, \cdots x_n) \cdot (g_1, \cdots g_n, \sigma) := (x_{\sigma(1)} g_{\sigma(1)}, \cdots x_{\sigma(n)} g_{\sigma(n)}).$

Definition of Equivariant Power Operation [May][Ganter]

$$P_n : E_G(X) \longrightarrow E_{G \wr \Sigma_n}(X^{\times n})$$

satisfying some axioms.

Quasi-elliptic cohomology has power operations

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X/\!\!/G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

The Elliptic Power Operation [Huan]

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)}:$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}).$$

$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\Lambda}_{k}} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

$$\xrightarrow{\boxtimes} K_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)})$$

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X /\!\!/ G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G \wr \Sigma_n}(X^{\times n})$$

The Elliptic Power Operation [Huan]

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in (G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)}:$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in (G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}).$$

$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\wedge}_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

$$\xrightarrow{\boxtimes} K_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)})$$

Quasi-elliptic cohomology has power operations

Atiyah's Power Operation [Ganter]

V: a vector bundle over $\Lambda(X /\!\!/ G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]} n}$ defines an operation

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

The Elliptic Power Operation [Huan]

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)}:$$

$$QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}) = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)}).$$

$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\wedge}_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$

$$\xrightarrow{\boxtimes} K_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)})$$

The Power Operation $\mathbb{P}_n$

Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.
When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

The Power Operation $\mathbb{P}_n$

Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.
When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

The Power Operation $\mathbb{P}_n$

Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.
When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.
When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

The Power Operation $\mathbb{P}_n$

Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.

When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

The Power Operation $\mathbb{P}_n$

Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

Example ($G = e$)

$QEll_G^*(X) = K_{\mathbb{T}}^*(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.
When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

Why is $\mathbb{P}_n$ good?

- The construction can be generalized to other cohomology theories.
- Uniquely extends to the **stringy power operation** of Tate K-theory.
- Elliptic: reflect the geometric structure of Tate curve.

Example ($G = e$)

$QEll^*_G(X) = K^*_{\mathbb{T}}(X)$. For each $\sigma \in \Sigma_n$, $\mathbb{P}_{(\underline{1},\sigma)}(x) = \boxtimes_k \boxtimes_{(i_1,\cdots i_k)} (x)_k$.
When $n = 2$,

$$\mathbb{P}_2(x) = (\mathbb{P}_{(\underline{1},(1)(1))}(x), \mathbb{P}_{(\underline{1},(12))}(x)) = (x \boxtimes x, (x)_2).$$

When $n = 3$, $\mathbb{P}_3(x) = (\mathbb{P}_{(\underline{1},(1)(1)(1))}(x), \mathbb{P}_{(\underline{1},(12)(1))}(x), \mathbb{P}_{(\underline{1},(123))}(x)) = (x \boxtimes x \boxtimes x, (x)_2 \boxtimes x, (x)_3)$.

Tate Curve and Representation Theory

Theorem (Huan)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[{q'_s}^{\pm}]/\langle q^d - {q'_s}^e\rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q'_s is the image of q under the stringy power operation, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

The proof: use representation theory via quasi-elliptic cohomology.

Theorem (Huan)

A: any finite abelian group.
$K_{Tate}(pt/\!\!/A)/I_{tr}^{A}$ classifies $A-$Level structure of the Tate curve.

Theorem (Huan)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[{q'_s}^{\pm}]/\langle q^d - {q'_s}^{e}\rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q'_s is the image of q under the stringy power operation, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

The proof: use representation theory via quasi-elliptic cohomology.

Theorem (Huan)

A: any finite abelian group.
$K_{Tate}(pt/\!\!/A)/I_{tr}^{A}$ classifies $A-$Level structure of the Tate curve.

Tate Curve and Representation Theory

Theorem (Huan)

The Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate} \cong \prod_{N=de} \mathbb{Z}((q))[{q'_s}^{\pm}]/\langle q^d - {q'_s}^e \rangle,$$

where I_{tr}^{Tate} is the transfer ideal and q'_s is the image of q under the stringy power operation, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

The proof: use representation theory via quasi-elliptic cohomology.

Theorem (Huan)

A: any finite abelian group.
$K_{Tate}(pt/\!\!/A)/I_{tr}^{A}$ classifies $A-$Level structure of the Tate curve.

• 1967 Daniel G. Quillen "Homotopical algebra": form the foundation of homotopy theory.
• 1997 Vladimir Voevodsky "The Milnor Conjecture": apply homotopical thinking to non-homotopical situation.

- 1967 Daniel G. Quillen "Homotopical algebra": form the foundation of homotopy theory.
- 1997 Vladimir Voevodsky "The Milnor Conjecture": apply homotopical thinking to non-homotopical situation.

• 1967 Daniel G. Quillen "Homotopical algebra": form the foundation of homotopy theory.
• 1997 Vladimir Voevodsky "The Milnor Conjecture": apply homotopical thinking to non-homotopical situation.

• 1967 Daniel G. Quillen "Homotopical algebra": form the foundation of homotopy theory.

• 1997 Vladimir Voevodsky "The Milnor Conjecture": apply homotopical thinking to non-homotopical situation.

Motivating example: Quillen model structure on topological spaces

Represent the standard homotopy theory of CW-complexes.

- weak homotopy equivalence
- Serre fibration
- retract of relative cell complex

• 1967 Daniel G. Quillen "Homotopical algebra": form the foundation of homotopy theory.
• 1997 Vladimir Voevodsky "The Milnor Conjecture": apply homotopical thinking to non-homotopical situation.

Definition: A model structure on a category $\mathcal{C}$

Three distinguished classes of morphisms:

- Weak Equivalence
- Fibration
- Cofibration

satisfying the axioms: • Retracts • 2 of 3 • Lifting • Factorization

• 1967 Daniel G. Quillen "Homotopical algebra": form the foundation of homotopy theory.

• 1997 Vladimir Voevodsky "The Milnor Conjecture": apply homotopical thinking to non-homotopical situation.

Definition: A model structure on a category $\mathcal{C}$

Three distinguished classes of morphisms:

- Weak Equivalence
- Fibration
- Cofibration

satisfying the axioms: • Retracts • 2 of 3 • Lifting • Factorization

Morphisms

Quillen adjunction: $(L \dashv R) : \mathcal{C} \overset{R}{\underset{L}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \overset{\mathbb{R}}{\underset{\mathbb{L}}{\leftrightarrows}} Ho(\mathcal{D})$.

Why do we care equivariant orthogonal spectra?

Right Philosophy for Stable Homotopy Theory (1990s) [MMSS]

$$\begin{tikzcd}
S-\text{Modules} \arrow[dr, equal, "Q.E."] & & \text{Symmetric Spectra} \arrow[dl, equal, "Q.E."'] \\
& \text{Spectra} & \\
\text{Orthogonal Spectra} \arrow[ur, equal, "Q.E."] & & \text{Many Others} \arrow[ul, equal, "Q.E."']
\end{tikzcd}$$

Why do we care equivariant orthogonal spectra?

Equivariant Stable Homotopy Theory (2000s) [MMSS]

$$\begin{tikzcd}[column sep=small]
S_G - \text{Modules} \arrow[dr, equal, "Q.E."] & & \text{Orthogonal G-Spectra} \arrow[dl, equal, "Q.E."] \\
& \text{G-Spectra} &
\end{tikzcd}$$

Why do we care equivariant orthogonal spectra?

Equivariant Stable Homotopy Theory (2000s) [MMSS]

$$\begin{tikzcd}
S_G - \text{Modules} \arrow[dr, equal, "Q.E."] & & \text{Orthogonal G-Spectra} \arrow[dl, equal, "Q.E."] \\
& \text{G-Spectra} &
\end{tikzcd}$$

Which is the BEST model?

Orthogonal $G-$spectra.

Why BEST?

Combine the best feature of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

Why do we care equivariant orthogonal spectra?

Equivariant Stable Homotopy Theory (2000s) [MMSS]

$$\begin{tikzcd}
S_G - \text{Modules} \arrow[dr, equal, "Q.E."] & & \text{Orthogonal G-Spectra} \arrow[dl, equal, "Q.E."] \\
& \text{G-Spectra} &
\end{tikzcd}$$

Which is the BEST model?

Orthogonal G−spectra.

Why BEST?

Combine the best feature of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

Why do we care equivariant orthogonal spectra?

Equivariant Stable Homotopy Theory (2000s) [MMSS]

$$\begin{tikzcd}
S_G - \text{Modules} \arrow[dr, equal, "Q.E."] & & \text{Orthogonal G-Spectra} \arrow[dl, equal, "Q.E."] \\
& \text{G-Spectra} &
\end{tikzcd}$$

Which is the BEST model?

Orthogonal $G-$spectra.

Why BEST?

Combine the best feature of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

Why do we care equivariant orthogonal spectra?

Equivariant Stable Homotopy Theory (2000s) [MMSS]

$$\begin{tikzcd}[column sep=small]
S_G - \text{Modules} \arrow[dr, equal, "Q.E."] & & \text{Orthogonal G-Spectra} \arrow[dl, equal, "Q.E."] \\
& \text{G-Spectra} &
\end{tikzcd}$$

Which is the BEST model?

Orthogonal $G-$spectra.

Why BEST?

Combine the best feature of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

- a unit $\eta : S \longrightarrow X$;
- a natural product $\mu : X \overline{\wedge} X \longrightarrow X \circ \bigoplus$

such that the evident unit, associativity and centrality of unit diagram commute.

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^- \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

- a unit $\eta : S \longrightarrow X$;
- a natural product $\mu : X\overline{\wedge}X \longrightarrow X \circ \bigoplus$

such that the evident unit, associativity and centrality of unit diagram commute.

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

- a unit $\eta : S \longrightarrow X$;
- a natural product $\mu : X \overline{\wedge} X \longrightarrow X \circ \bigoplus$

such that the evident unit, associativity and centrality of unit diagram commute.

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^- \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

- a unit $\eta : S \longrightarrow X$;
- a natural product $\mu : X \overline{\wedge} X \longrightarrow X \circ \bigoplus$

such that the evident unit, associativity and centrality of unit diagram commute.

Global Homotopy Theory [Schwede][May]

Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

$\Rightarrow$ **global homotopy theory**

Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

Orthogonal Space [Schwede]

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An orthogonal space is a continuous functor $Y : \mathbb{L} \longrightarrow \mathcal{T}$ to the category of topological spaces.

Global Homotopy Theory [Schwede][May]

Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.
$\Rightarrow$ **global homotopy theory**

Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

Orthogonal Space [Schwede]

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An orthogonal space is a continuous functor $Y : \mathbb{L} \longrightarrow \mathcal{T}$ to the category of topological spaces.

Global Homotopy Theory [Schwede][May]

Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.
$\Rightarrow$ **global homotopy theory**
Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

Orthogonal Space [Schwede]

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An orthogonal space is a continuous functor $Y : \mathbb{L} \longrightarrow \mathcal{T}$ to the category of topological spaces.

Global Homotopy Theory [Schwede][May]

Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.
$\Rightarrow$ **global homotopy theory**
Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

Orthogonal Space [Schwede]

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An orthogonal space is a continuous functor $Y : \mathbb{L} \longrightarrow \mathcal{T}$ to the category of topological spaces.

Global Homotopy Theory

Global Homotopy Theory [Schwede][May]

Observation: It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.
$\Rightarrow$ **global homotopy theory**
Prominent examples: equivariant stable homotopy, equivariant K-theory, equivariant bordism.

Orthogonal Space [Schwede]

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An orthogonal space is a continuous functor $Y : \mathbb{L} \longrightarrow \mathcal{T}$ to the category of topological spaces.

an immediate idea to construct a spectrum $\{X_n, \psi_n\}_n$

$\{KU_{G,n}, \phi_n\}_n$: a $G-$spectrum representing $K^*_G(-)$.

If $X \mapsto X^\sigma$ has a right adjoint r_σ,

$$X_n = \prod_{\sigma \in G^{tors}_{conj}} r_\sigma(KU_{\Lambda_G(\sigma),n}), \ \ \psi_n = \prod_{\sigma \in G^{tors}_{conj}} r_\sigma(\phi_n).$$

However, r_σ does not exist.

an immediate idea to construct a spectrum $\{X_n, \psi_n\}_n$

$\{KU_{G,n}, \phi_n\}_n$: a $G-$spectrum representing $K^*_G(-)$.
If $X \mapsto X^\sigma$ has a right adjoint r_σ,

$$X_n = \prod_{\sigma \in G^{tors}_{conj}} r_\sigma(KU_{\Lambda_G(\sigma),n}), \;\; \psi_n = \prod_{\sigma \in G^{tors}_{conj}} r_\sigma(\phi_n).$$

However, r_σ does not exist.

Orthogonal $G-$spectrum of Quasi-elliptic cohomology

an immediate idea to construct a spectrum $\{X_n, \psi_n\}_n$

$\{KU_{G,n}, \phi_n\}_n$: a $G-$spectrum representing $K^*_G(-)$.
If $X \mapsto X^\sigma$ has a right adjoint r_σ,

$$X_n = \prod_{\sigma \in G^{tors}_{conj}} r_\sigma(KU_{\Lambda_G(\sigma),n}), \;\; \psi_n = \prod_{\sigma \in G^{tors}_{conj}} r_\sigma(\phi_n).$$

However, r_σ does not exist.

Homotopical Adjunction [Rezk]

H, G: compact Lie groups. X: $G-$CW complex.

$$\mathrm{Map}_H(LX, Y) \simeq \mathrm{Map}_G(X, RY).$$

Homotopical Adjunction [Rezk]

H, G: compact Lie groups. X: $G-$CW complex.

$$\mathrm{Map}_H(LX, Y) \simeq \mathrm{Map}_G(X, RY).$$

Homotopical right adjoint $\mathcal{R}_\sigma$ exists. I construct one!

Orthogonal $G-$spectrum of Quasi-elliptic cohomology

Homotopical Adjunction [Rezk]

H, G: compact Lie groups. X: $G-$CW complex.

$$\mathrm{Map}_H(LX, Y) \simeq \mathrm{Map}_G(X, RY).$$

Homotopical right adjoint $\mathcal{R}_\sigma$ exists. I construct one!

GwS: the new category of orthognal $G-$spectra [Huan]

GwS := the homotopy category of the category of orthogonal $G-$spectra with the weak equivalence defined by

$$X \sim Y \text{ if } \pi_0(X(V)) = \pi_0(Y(V)),$$

for each faithful $G-$representation V.

An orthogonal $G-$spectrum X in GwS is said to represent a theory H^*_G if we have a natural map

$$\pi_0(X(V)) = H^V_G(S^0),$$

for each faithful $G-$representation V.

Orthogonal $G-$spectrum of Quasi-elliptic cohomology

Theorem [Huan]

There exists a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$ representing $QEll_G^*(-)$ in GwS.

The construction can be generalized. [Huan]

There is a well-defined functor from the full subcategory consisting of $\mathcal{I}_G-$FSP in GwS to itself.

Can $E(G,-)$ arise from an orthogonal spectrum?

Orthogonal $G-$spectrum of Quasi-elliptic cohomology

Theorem [Huan]

There exists a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$ representing $QEll^*_G(-)$ in GwS.

The construction can be generalized. [Huan]

There is a well-defined functor from the full subcategory consisting of $\mathcal{I}_G-$FSP in GwS to itself.

Can $E(G,-)$ arise from an orthogonal spectrum?

Orthogonal $G-$spectrum of Quasi-elliptic cohomology

Theorem [Huan]

There exists a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$ representing $QEll^*_G(-)$ in GwS.

The construction can be generalized. [Huan]

There is a well-defined functor from the full subcategory consisting of $\mathcal{I}_G-$FSP in GwS to itself.

Can $E(G,-)$ arise from an orthogonal spectrum?

Theorem [Huan]

There exists a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$ representing $QEll_G^*(-)$ in GwS.

The construction can be generalized. [Huan]

There is a well-defined functor from the full subcategory consisting of $\mathcal{I}_G-$FSP in GwS to itself.

Can $E(G,-)$ arise from an orthogonal spectrum?

$$\text{Orthogonal spectra} \underset{\text{arise?}}{\overset{u}{\rightleftarrows}} \text{Equivariant orthogonal spectra}$$

Orthogonal $G-$spectrum of Quasi-elliptic cohomology

Theorem [Huan]

There exists a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$ representing $QEll_G^*(-)$ in GwS.

The construction can be generalized. [Huan]

There is a well-defined functor from the full subcategory consisting of $\mathcal{I}_G-$FSP in GwS to itself.

Can $E(G,-)$ arise from an orthogonal spectrum?

$$\text{Orthogonal spectra} \underset{\text{arise?}}{\overset{u}{\rightleftarrows}} \text{Equivariant orthogonal spectra}$$

- u: $X \mapsto X\langle G\rangle$ underlying orthogonal $G-$spectrum;
- arise: yes iff for any trivial $G-$representation V, the $G-$action on $Y(V)$ is trivial.

Orthogonal $G-$spectrum of Quasi-elliptic cohomology

Theorem [Huan]

There exists a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$ representing $QEll_G^*(-)$ in GwS.

The construction can be generalized. [Huan]

There is a well-defined functor from the full subcategory consisting of $\mathcal{I}_G-$FSP in GwS to itself.

Can $E(G,-)$ arise from an orthogonal spectrum?

No.

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not trivial.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Improve the current theory

- Examples are limited.
- The restriction maps are identity.
- Almost impossible to construct global elliptic cohomology theory.

The idea: we still use diagram spectra to construct it.

The category D_0: add restriction maps to $\mathbb{L}$

This is a generalized Reedy category.

- Objects: (G, V);
- Morphisms:
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can define global quasi-elliptic cohomology.

Combining the equivariant orthogonal spectra $\{E(G, -)\}$, we get an ultra-commutative global ring spectrum in the new theory.

Model Structure: How to view the new theory

We formulate several model structures.

- Level Model Structure;
- Generalized Reedy Model Structure.

Conjecture [Huan]

There is a global model structure on the almost global spaces that is Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

Model Structure: How to view the new theory

We formulate several model structures.

- Level Model Structure;
- Generalized Reedy Model Structure.

Conjecture [Huan]

There is a global model structure on the almost global spaces that is Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.

Further problems: Loop space and Physics

Generalization to elliptic cohomology theories

- A young conjecture (mine): the relation between elliptic cohomology theories and loop space can be interpreted by bibundles.
- An old conjecture: the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve (and that on Morava E-theories).

Relation between quasi-elliptic cohomology and physics. (Plan)

- Express quasi-elliptic cohomology by group schemes based on Lurie's idea.
- Construct physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology.

Further problems: Loop space and Physics

Generalization to elliptic cohomology theories

- A young conjecture (mine): the relation between elliptic cohomology theories and loop space can be interpreted by bibundles.
- An old conjecture: the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve (and that on Morava E-theories).

Relation between quasi-elliptic cohomology and physics. (Plan)

- Express quasi-elliptic cohomology by group schemes based on Lurie's idea.
- Construct physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology.

Further problems: Loop space and Physics

Generalization to elliptic cohomology theories

- A young conjecture (mine): the relation between elliptic cohomology theories and loop space can be interpreted by bibundles.
- An old conjecture: the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve (and that on Morava E-theories).

Relation between quasi-elliptic cohomology and physics. (Plan)

- Express quasi-elliptic cohomology by group schemes based on Lurie's idea.
- Construct physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology.

Further problems: Loop space and Physics

Generalization to elliptic cohomology theories

- A young conjecture (mine): the relation between elliptic cohomology theories and loop space can be interpreted by bibundles.
- An old conjecture: the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve (and that on Morava E-theories).

Relation between quasi-elliptic cohomology and physics. (Plan)

- Express quasi-elliptic cohomology by group schemes based on Lurie's idea.
- Construct physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology.

Further problems: Loop space and Physics

Generalization to elliptic cohomology theories

- A young conjecture (mine): the relation between elliptic cohomology theories and loop space can be interpreted by bibundles.
- An old conjecture: the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve (and that on Morava E-theories).

Relation between quasi-elliptic cohomology and physics. (Plan)

- Express quasi-elliptic cohomology by group schemes based on Lurie's idea.
- Construct physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology.

Further problems: Loop space and Physics

Generalization to elliptic cohomology theories

- A young conjecture (mine): the relation between elliptic cohomology theories and loop space can be interpreted by bibundles.
- An old conjecture: the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve (and that on Morava E-theories).

Relation between quasi-elliptic cohomology and physics. (Plan)

- Express quasi-elliptic cohomology by group schemes based on Lurie's idea.
- Construct physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology.

Further problems: equivariant and global spectrum

Representing object of equivariant elliptic cohomology

Chenchang Zhu suggests, other than Gepner's and my approach, we can use the geometric object two-vector bundles representing elliptic cohomology to establish equivariant elliptic cohomology theories.

Global homotopy theory for elliptic cohomology [Huan]

Quasi-elliptic cohomology serves as a breakthrough point of the construction.

- Continue with the global homotopy theory under construction: construct the global model structure and the corresponding stable global homotopy theory.
- Another possibility: Gepner and Rezk's global homotopy theory via orbispaces and infinite category theory.
  - What's the relation between the two global homotopy theories?
  - My conjecture: Gepner and Rezk' global homotopy theory is more suitable for constructing global elliptic cohomology theory.

Further problems: equivariant and global spectrum

Representing object of equivariant elliptic cohomology

Chenchang Zhu suggests, other than Gepner's and my approach, we can use the geometric object two-vector bundles representing elliptic cohomology to establish equivariant elliptic cohomology theories.

Global homotopy theory for elliptic cohomology [Huan]

Quasi-elliptic cohomology serves as a breakthrough point of the construction.

- Continue with the global homotopy theory under construction: construct the global model structure and the corresponding stable global homotopy theory.
- Another possibility: Gepner and Rezk's global homotopy theory via orbispaces and infinite category theory.
  - What's the relation between the two global homotopy theories?
  - My conjecture: Gepner and Rezk' global homotopy theory is more suitable for constructing global elliptic cohomology theory.

Further problems: equivariant and global spectrum

Representing object of equivariant elliptic cohomology

Chenchang Zhu suggests, other than Gepner's and my approach, we can use the geometric object two-vector bundles representing elliptic cohomology to establish equivariant elliptic cohomology theories.

Global homotopy theory for elliptic cohomology [Huan]

Quasi-elliptic cohomology serves as a breakthrough point of the construction.

- Continue with the global homotopy theory under construction: construct the global model structure and the corresponding stable global homotopy theory.
- Another possibility: Gepner and Rezk's global homotopy theory via orbispaces and infinite category theory.
  - What's the relation between the two global homotopy theories?
  - My conjecture: Gepner and Rezk' global homotopy theory is more suitable for constructing global elliptic cohomology theory.

Further problems: equivariant and global spectrum

Representing object of equivariant elliptic cohomology

Chenchang Zhu suggests, other than Gepner's and my approach, we can use the geometric object two-vector bundles representing elliptic cohomology to establish equivariant elliptic cohomology theories.

Global homotopy theory for elliptic cohomology [Huan]

Quasi-elliptic cohomology serves as a breakthrough point of the construction.

- Continue with the global homotopy theory under construction: construct the global model structure and the corresponding stable global homotopy theory.
- Another possibility: Gepner and Rezk's global homotopy theory via orbispaces and infinite category theory.
  - What's the relation between the two global homotopy theories?
  - My conjecture: Gepner and Rezk' global homotopy theory is more suitable for constructing global elliptic cohomology theory.

Further problems: equivariant and global spectrum

Representing object of equivariant elliptic cohomology

Chenchang Zhu suggests, other than Gepner's and my approach, we can use the geometric object two-vector bundles representing elliptic cohomology to establish equivariant elliptic cohomology theories.

Global homotopy theory for elliptic cohomology [Huan]

Quasi-elliptic cohomology serves as a breakthrough point of the construction.

- Continue with the global homotopy theory under construction: construct the global model structure and the corresponding stable global homotopy theory.
- Another possibility: Gepner and Rezk's global homotopy theory via orbispaces and infinite category theory.
  - What's the relation between the two global homotopy theories?
  - My conjecture: Gepner and Rezk' global homotopy theory is more suitable for constructing global elliptic cohomology theory.

Further problem: Hopkins-Kuhn-Ravenel character theory

Hopkins, Kuhn, Ravenel have shown that Devoto's equivariant K-theory has a Hopkins-Kuhn-Ravenel (HKR) character theory.

Ganter's conjecture: the HKR character theory for orbifold Tate K-theory to be established via **Stapleton**'s framework of tanschromatic character theory.

My conjecture: the HKR character theory for quasi-elliptic cohomology exists and can probably be fit into **Stapleton**'s transchromatic character theory.

Further problem: Hopkins-Kuhn-Ravenel character theory

Hopkins, Kuhn, Ravenel have shown that Devoto's equivariant K-theory has a Hopkins-Kuhn-Ravenel (HKR) character theory.

Ganter's conjecture: the HKR character theory for orbifold Tate K-theory to be established via **Stapleton**'s framework of tanschromatic character theory.

My conjecture: the HKR character theory for quasi-elliptic cohomology exists and can probably be fit into **Stapleton**'s transchromatic character theory.

Hopkins, Kuhn, Ravenel have shown that Devoto's equivariant K-theory has a Hopkins-Kuhn-Ravenel (HKR) character theory.

Ganter's conjecture: the HKR character theory for orbifold Tate K-theory to be established via **Stapleton**'s framework of tanschromatic character theory.

My conjecture: the HKR character theory for quasi-elliptic cohomology exists and can probably be fit into **Stapleton**'s transchromatic character theory.

Further problem: Hopkins-Kuhn-Ravenel character theory

Hopkins, Kuhn, Ravenel have shown that Devoto's equivariant K-theory has a Hopkins-Kuhn-Ravenel (HKR) character theory.

Ganter's conjecture: the HKR character theory for orbifold Tate K-theory to be established via **Stapleton**'s framework of tanschromatic character theory.

My conjecture: the HKR character theory for quasi-elliptic cohomology exists and can probably be fit into **Stapleton**'s transchromatic character theory.

$$QEll_G(X) := K_{orb}(\Lambda(X/\!\!/G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X/\!\!/G)$: constructed via the idea of $\Lambda(X/\!\!/G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma \in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- ······

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X /\!\!/ G)$: constructed via the idea of $\Lambda(X /\!\!/ G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma \in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- $\cdots\cdots$

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g\in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X /\!\!/ G)$: constructed via the idea of $\Lambda(X /\!\!/ G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma\in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- ······

$$QEll_G(X) := K_{orb}(\Lambda(X/\!\!/G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X/\!\!/G)$: constructed via the idea of $\Lambda(X/\!\!/G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma \in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- ······

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X /\!\!/ G)$: constructed via the idea of $\Lambda(X /\!\!/ G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma \in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- $\cdots\cdots$

$$QEll_G(X) := K_{orb}(\Lambda(X/\!\!/G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X/\!\!/G)$: constructed via the idea of $\Lambda(X/\!\!/G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma \in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- ······

$$QEll_G(X) := K_{orb}(\Lambda(X/\!\!/G)) \cong \prod_{g\in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X/\!\!/G)$: constructed via the idea of $\Lambda(X/\!\!/G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma\in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- ······

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X /\!\!/ G)$: constructed via the idea of $\Lambda(X /\!\!/ G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma \in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- $\cdots\cdots$

$$QEll_G(X) := K_{orb}(\Lambda(X /\!\!/ G)) \cong \prod_{g \in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

- E^*: any equivariant cohomology theory.
- $\Lambda^n(X /\!\!/ G)$: constructed via the idea of $\Lambda(X /\!\!/ G)$, analogous to the topological groupoid $Fix_n(X)$ in [HKR].

$$\mathbf{E}^*(X) := \prod_{\sigma \in G^n_z} E_{\Lambda^n(\sigma)}(X^\sigma).$$

- Features?
- Power operations?
- Equivariant spectrum?
- fit into the new global homotopy theory?
- ······

My Last Question

Can we construct an intermediate theory for each elliptic cohomology theory which

- reflects the geometric nature of elliptic curves
- more practicable to study?

My Last Question

Can we construct an intermediate theory for each elliptic cohomology theory which

- reflects the geometric nature of elliptic curves
- more practicable to study?

I hope so.

Thank you.

Some references

http://gagp.sysu.edu.cn/zhenhuan/Zhen-CAS-2018-Slides.pdf

- Ando, “Isogenies of formal group laws and power operations in the cohomology theories E_n”, Duke J., 1995
- Ando, Hopkins, Strickland: "Elliptic spectra, the Witten genus and the theorem of the cube". Invent. Math. , 146(3):595–687, 2001.
- Ando, Hopkins, Strickland, “The sigma orientation is an H_∞ map”, Amer. J. 2004;
- Atiyah, "Power operations in K-theory", Quart. J. Math. Oxford Ser. (2) 17 1966.
- Baas, Dundas, Rognes: "Two-vector bundles and forms of elliptic cohomology", Topology, geometry and quantum field theory, 18–45, London Math. Soc. Lecture Note Ser., 308, Cambridge Univ. Press, Cambridge, 2004.
- Berger, Moerdijk, "On an extension of the notion of Reedy category", Mathematische Zeitschrift, December 2011.
- Devoto, "Equivariant elliptic homology and finite groups", Michigan Math. J. , 43(1):3–32, 1996.
- Ganter, “Orbifold genera, product formulas, and power operations”, Adv. Math, 2006;
- Ganter, “Stringy power operations in Tate K-theory”, Homology, Homotopy, Appl., 2013;
- Gepner, "Homotopy Topoi and Equivariant Elliptic Cohomology", Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 1999.
- Ginzburg, Kapranov, Vasserot, "Elliptic algebras and equivariant elliptic cohomology I", available at arXiv:q-alg/9505012.
- Hopkins, Kuhn, Ravenel, "Generalized group characters and complex oriented cohomology theories", J. Am. Math. Soc. 13 (2000).
- Landweber, "Elliptic Curves and Modular Forms in Algebraic Topology: proceedings of a conference held at the Institute for Advanced Study", Princeton, September 1986, Lecture Notes in Mathematics.
- Lerman, "Orbifolds as stacks", Enseign. Math. (2) 56 (2010), no. 3–4, 315–363.
- Lurie, "A Survey of Elliptic Cohomology", in Algebraic Topology Abel Symposia Volume 4, 2009, pp 219–277.
- Mandell, May, Schwede, Shipley, "Model categories of diagram spectra", Proc. London Math. Soc. 82(2001).
- May, "Equivariant homotopy and cohomology theory", CBMS Regional Conference Series in Mathematics, vol. 91, 1996.
- Rezk, "Quasi-elliptic cohomology", 2011.
- Schwede, "Global Homotopy Theory", `global.pdf`.
- Tomer M. Schlank, Nathaniel Stapleton, "A transchromatic proof of Strickland's theorem", Adv. Math. 285 (2015), 1415–1447.
- Stapleton, "Transchromatic generalized character maps", Algebr. Geom. Topol. 13 (2013), no. 1, 171–203.
- Strickland, "Morava E-theory of symmetric groups", Topology 37 (1998), no. 4.