Quasi-Elliptic Cohomology

Zhen Huan

Sun Yat-sen University

International Festival in Schubert Calculus, November 6-10, 2017

Power Operation

Quasi-elliptic cohomology has power operations. So it has the structure of an "$H_\infty-$ring theory".

Atiyah's Power Operation as Equivariant K-theories

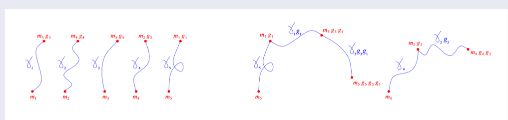
V: a vector bundle over $\Lambda(X/\!\!/G)$.

$$P_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n}),\ V \mapsto V^{\widehat{\otimes}_{\mathbb{Z}[q^\pm]} n}$$

The Power Operation that is "Elliptic"

$$\mathbb{P}_n = \prod_{(\underline{g},\sigma)\in(G\wr\Sigma_n)^{tors}_{conj}} \mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

$$\mathbb{P}_{(\underline{g},\sigma)} : QEll_G(X) \xrightarrow{U^*} K_{orb}(\Lambda_{(\underline{g},\sigma)}(X)) \xrightarrow{(\ )^{\wedge}_k} K_{orb}(\Lambda^{var}_{(\underline{g},\sigma)}(X))$$
$$\xrightarrow{\boxtimes} K_{orb}(d_{(\underline{g},\sigma)}(X)) \xrightarrow{f^*_{(\underline{g},\sigma)}} K_{\Lambda_{G\wr\Sigma_n}(\underline{g},\sigma)}((X^{\times n})^{(\underline{g},\sigma)})$$



Question

Can we have Strickland's theorem for Tate K-Theory?

The Finite Subgroups of the Tate Curve

The finite subgroups of the Tate curve of order N can be classified by $\mathcal{O}_{Sub_N} = \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle$.

Transfer Ideals

$$I^{Tate}_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[I^{\Sigma_N}_{\Sigma_i\times\Sigma_j}: K_{Tate}(\mathrm{pt}/\!\!/\Sigma_i\times\Sigma_j) \longrightarrow K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)]$$

$$\mathcal{I}^{QEll}_{tr} := \sum_{\substack{i+j=N,\\ N>j>0}} \mathrm{Image}[\mathcal{I}^{\Sigma_N}_{\Sigma_i\times\Sigma_j}: QEll(\mathrm{pt}/\!\!/\Sigma_i\times\Sigma_j) \longrightarrow QEll(\mathrm{pt}/\!\!/\Sigma_N)]$$

Classification of the Finite Subgroups of Tate Curve

Theorem (Huan)

$$QEll(pt/\!\!/\Sigma_N)/\mathcal{I}^{QEll}_{tr} \cong \prod_{N=de} \mathbb{Z}[q^{\pm}][q'^{\pm}]/\langle q^d - q'^e\rangle,$$

where q' is the image of q under the power operation $\mathbb{P}_N$.

Theorem (Huan)

The Tate K-theory of symmetric groups modulo the transfer ideal classifies the finite subgroups of the Tate curve.

$$K_{Tate}(pt/\!\!/\Sigma_N)/I^{Tate}_{tr} \cong \prod_{N=de} \mathbb{Z}((q))[q_s'^{\pm}]/\langle q^d - q_s'^{e}\rangle,$$

where q_s' is the image of q under the stringy power operation P^{string}_N.

Motivation: An old idea by Witten

- The elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops.
- It's surprisingly difficult to make this precise!
- Why? In application, one needs to consider the case that a group G acts on X. In this case the loop space LX has rich structures as an orbifold.

The relation between Tate K-theory and the loop space $\Rightarrow$ quasi-elliptic cohomology.

Construction

The Key Concept: Bibundles

A bibundle from $\mathbb{H}$ to $\mathbb{G}$ is a smooth manifold P together with

- the structure maps:
  - $\tau : P \longrightarrow \mathbb{G}_0$;
  - a surjective submersion $\sigma : P \longrightarrow \mathbb{H}_0$.
- The action maps in $Man_{G_0\times H_0}$
  - $\mathbb{G}_1\ {}_s\!\times_\tau P \longrightarrow P$;
  - $P\ {}_\sigma\!\times_t \mathbb{H}_1 \longrightarrow P$

such that

- $g_1\cdot(g_2\cdot p) = (g_1g_2)\cdot p$; • $(p\cdot h_1)\cdot h_2 = p\cdot(h_1h_2)$; • $g\cdot(p\cdot h) = (g\cdot p)\cdot h$
- $p\cdot u_H(\sigma(p)) = p$ and $u_G(\tau(p))\cdot p = p$ for all $p\in P$.
- $\mathbb{G}_1\ {}_s\!\times_\tau P \longrightarrow P\ {}_\sigma\!\times_\sigma P$ $(g,p)\mapsto(g\cdot p, p)$ is an isomorphism.

The Loop Space of Interest: $Loop(X/\!\!/G) := Bibun(S^1/\!\!/*, X/\!\!/G)$

- Objects:

$$\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$$

  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;
- Morphism $\mathcal{P}\longrightarrow\mathcal{P}'$: $G-$bundle map $\alpha : P \longrightarrow P'$

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ & \searrow^{\pi'} & \downarrow{\alpha} & \nearrow_{f'} & \\ & & P' & & \end{array}$$

The Loop Space with All the Data: $Loop^{ext}(X/\!\!/G)$

Add the rotations:

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ \downarrow{t} & & \downarrow{\alpha} & \nearrow_{f'} & \\ S^1 & \xleftarrow{\pi'} & P' & & \end{array}$$

The Groupoid We Really Study: $\Lambda(X/\!\!/G)$

A subgroupoid of $Loop^{ext}(X/\!\!/G)$

$$\Lambda(X/\!\!/G) := \coprod_{g\in G^{tors}_{conj}} X^g/\!\!/\Lambda_G(g)$$

G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors};

$$\Lambda_G(g) = C_G(g)\times\mathbb{R}/\langle(g,-1)\rangle$$

The Definition of Quasi-elliptic cohomology

$$QEll_G(X) := K_{orb}(\Lambda(X/\!\!/G)) \cong \prod_{g\in G^{tors}_{conj}} K_{\Lambda_G(g)}(X^g)$$

Relation with Tate K-theory

$$QEll^*_G(X)\otimes_{\mathbb{Z}[q^\pm]}\mathbb{Z}((q)) \cong K^*_{Tate}(X/\!\!/G).$$

The Spectra of Equivariant Elliptic Cohomologies

Ginzburg, Kapranov and Vasserot's Conjecture

Any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A.

We get some idea with the construction of the orthogonal $G-$spectrum of quasi-elliptic cohomology.

Orthogonal $G-$spectrum of quasi-elliptic cohomology

We construct explicitly a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$. For each faithful $G-$representation V, $E(G,V)$ weakly represents $QEll^V_G(-)$ in the sense

$$\pi_k(E(G,V)) = QEll^V_G(S^k), \text{ for each } k.$$

The construction can be applied to generalized Morava E-theory and equivariant Tate K-theory.

Can $E(G,-)$ arise from an orthogonal spectrum?

No.

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not trivial.

Then, it's even more difficult for equivariant elliptic cohomology to fit into the global homotopy theory!

Solution: Construct a New Global Homotopy Theory

Instead of orthogonal spaces, we study D_0-spaces.

The Category D_0

- objects: (G,V,ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations $\rho : G\longrightarrow O(V)$,
- morphism: $\phi = (\phi_1,\phi_2) : (G,V,\rho)\longrightarrow(H,W,\tau)$
  - $\phi_2 : V\longrightarrow W$ a linear isometric embedding
  - $\phi_1 : \tau^{-1}(O(\phi_2(V)))\longrightarrow G$ group homomorphism

$$\begin{array}{ccc} G & \xrightarrow{\rho} & O(V) \\ \uparrow{\phi_1} & & \downarrow{\phi_{2*}} \\ \tau^{-1}(O(\phi_2(V))) & \xrightarrow{\tau} & O(W) \end{array}$$

Features of the New Global Homotopy Theory

- An extension of global homotopy theory;
- Classifies those theories that should be global;
- The restriction maps are equivariant weak equivalence;
- We can define global quasi-elliptic cohomology by combining the orthogonal $G-$spectrum $\{E(G,-)\}$;
- The new and old global homotopy theories describe the same mathematical world. We are proving

Conjecture

There is a global model structure on the almost global spaces that is Quillen equivalent to the global model structure on the orthogonal spaces formulated by Schwede in Global Homotopy Theory.