RESEARCH STATEMENT

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The main subject that I'm studying is quasi-elliptic cohomology. It is very close to elliptic cohomology theories and thus can reflect the geometric nature of elliptic curves. The theory can be expressed in term of equivariant K-theory. It relates to many important areas in mathematics, including homotopy theory, geometric representation theory, equivariant elliptic cohomology theory, algebraic geometry and mathematical physics.

I hope to visit Bonn during March 2020. In December I established collaboration with Matthew Young, a post-doctoral fellow at the Max Planck Institute for Mathematics in Bonn. We developed several research projects, involving Hopkins-Kuhn Ravenel Character theroy for $n-$groups, quasi-elliptic cohomology for $2-$groups, Real and twisted Chern character map from the corresponding quasi-elliptic cohomology theory. Moreover, we are formulating the definition of power operation for twisted equivariant cohomology theories, construct a reasonable power operation for twisted quasi-elliptic cohomology theory and that for twisted Morava E-theories in the future. During my visit in Bonn I can continue discussing with him and develop further projects based on our progress. Moreover, Nathaniel Stapleton, another collaborator of mine, will be in Bonn in March 2020. We can continue discussion on our projects on Morava E-theories, which is an intersection point of algebraic geometry and algebraic topology. On my side, this opportunity is essential. There are many interesting points in this field that I want to explore. In addition, I would like to communicate with Professor Stefan Schwede and improve my earlier work on global homotopy theory.

Comparing to elliptic cohomology theories, quasi-elliptic cohomology is easier to compute and it supports neater constructions. It has shown great power in solving several significant mathematical problems. It can be constructed as the orbifold K-theory of an orbifold loop space, which partially proved a conjecture by Witten. Moreover, we formulate the total power operation of it and prove that the Tate K-theory of symmetric groups modulo a certain transfer ideal classify the finite subgroups of the Tate curve. This result shows that a conjecture in elliptic cohomology theories is true for the generalized elliptic cohomology associated to the Tate curve. In addition, we construct equivariant orthogonal spectra representing quasi-elliptic cohomology. The idea and technique can be applied to a family of other theories, including generalized Morava E-theories and equivariant Tate K-theory. In addition, motivated by quasi-elliptic cohomology, we formulate a new global homotopy theory which is equivalent to Schwede's global homotopy theory. What is more fascinating is that the theory itself has distinct features worth exploration. It serves as a meaningful motivating example to the extension of the idea of representation theory and equivariant homotopy theory to many popular areas.

The concept of elliptic cohomology was first introduced by Landweber, Ravenel, Stong in 1986. An elliptic cohomology theory is an even periodic multiplicative generalized cohomology theory whose associated formal group is the formal completion of an elliptic curve. It is at the intersection of a variety of areas in mathematics, including algebraic topology, algebraic geometry, mathematical physics, representation theory and number theory. Definitions of equivariant elliptic cohomology from different perspectives have been motivated, Devoto's definition motivated by orbifold string theory [17], Grojnowski's definition motivated by the theory of loop groups [33], etc. Moreover, in 1995 Ginzburg, Kapranov and Vasserot gave an axiomatic definition in term of principal bundles over elliptic curves. In 2007 based on Lurie's definition of elliptic cohomology [48] Gepner formulated

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a construction of equivariant elliptic cohomology [28] satisfying axioms similar to those in [30] in the context of derived algebraic geometry.

The generalized elliptic curve that is most relevant to quasi-elliptic cohomology is the Tate curve [4]. It can be considered as the boundary point of the moduli space of elliptic curves. The generalized elliptic cohomology theory associated to the Tate curve is Tate K-theory [4], which is itself a distinctive subject to study. Its relation with string theory is better understood than most known elliptic cohomology theories. In addition, the definition of $G-$equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold [Section 2, [24]].

Quasi-elliptic cohomology is not an elliptic cohomology but it contains all the information of equivariant Tate K-theory. That's how it got its name. It was first introduced by Ganter inspired by Devoto's equivariant Tate K-theory [17]. It is presented originally in Rezk's unpublished manuscript [52] and in full detail in my paper [38] and [36].

Moreover, Spong and I construct a Chern character map from quasi-elliptic cohomology theory to Devoto's equivariant elliptic cohomology theory. Moreover, we define twisted quasi-elliptic cohomology. To provide a geometric interpretation of it, we define a version of twisted equivariant loop space via bibundles. In addition, we construct a twisted Chern character map from it to twisted equivariant elliptic cohomology theory.

In addition, I'm exploring the nature of Morava E-theory. It is built out of the Lubin-Tate ring associated to a finite height formal group over a perfect field of characteristic p by using the Landweber exact functor theorem. It is a generalization of $p-$adic K-theory to higher height. Thus the E-cohomology of a finite group is a natural generalization of the representation ring at the prime p of the group. Hopkins, Kuhn, and Ravenel made signi?cant progress towards understanding these rings by constructing an analogue of the character map in representation theory for them[34]. In addition, at height 2 it is a form of elliptic cohomology closely related to Tate K-theory. Thus, the study of it and elliptic cohomology can benefit each other. Moreover, Lubin-Tate theory plays an important role in local arithmetic geometry. Thus, we can expect important objects in arithmetic geometry, such as the Drinfeld ring. Stapleton and I studied the level structure [19, Section 4] of the generalized Morava E-theory, which is a loopy extension $E^0(\mathcal{L}^k(-))$ of Morava E-theory. We study $E^0(\mathcal{L}^k(BA))/I_{tr}$ for A finite abelian. We prove a variety of results analogous to the more classical results regarding $E^0(BA)/I_{tr}$. Among other things, we study the product decomposition of $E^0(\mathcal{L}^k(BA))/I_{tr}$ by making use of certain families of subgroups of A, give an algebro-geometric description of these E^0-algebras in terms of level structures on $\mathbb{G}_E \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^k$, and describe the relation to $E^0(\mathcal{L}^k(B\Sigma_m))/I_{tr}$.

In Section 1, 2, 5 and 6 I present the contribution of quasi-elliptic cohomology to the study of elliptic cohomology theories, i.e. how we apply quasi-elliptic cohomology to study math problems on elliptic cohomology theory. In Section 3 I present our results on Chern character maps and twisted quasi-elliptic cohomology. In Section 4 I give an algebro-geometric description of $E^0(\mathcal{L}^k(BA))/I_{tr}$ via A^*-level structure. Based on our research progress so far, in Section 7 I list the research problems that we will work on in the near future.

1. Witten's conjecture

In [46] Landweber cited a conjecture by Witten that that the elliptic cohomology of a space X is related to the $\mathbb{T}-$equivariant K-theory of the free loop space $LX = \mathbb{C}^\infty(S^1, X)$ with the circle $\mathbb{T}$ acting on LX by rotating loops. However, the relation is usually difficult to interpret because in application we need to study a space X with a group action. In this case LX is an orbifold with sophisticated structure. In my PhD thesis [38] and [36] we construct quasi-elliptic cohomology as orbifold K-theory of a specific loop space. This construction gives an interpretation of equivariant Tate K-theory via loop spaces. It provides an approach other than Ganter's construction [Section 2, [24]].

If G is a Lie group and X is a manifold with a smooth $G-$action, the space of smooth unbased loops in the orbifold $X/\!\!/G$ in principle carries a lot of structure: on the one hand, it includes loops represented by continuous maps $\gamma : \mathbb{R} \longrightarrow X$ such that $\gamma(t+1) = \gamma(t)g$ for some $g \in G$; at the same time the circle acts on the loop space by rotation. We interpret a "loop" as a bibundle, i.e. a 1-morphism in the localization of the category of Lie groupoids with respect to the equivalence of Lie groupoids. The right loop space $Loop^{ext}(X/\!\!/G)$ is constructed from the category of bibundles from $S^1/\!\!/*$ to $X/\!\!/G$ with the circle rotations added as morphisms.

Quasi-elliptic cohomology $QEll^*_G(X)$ is defined to be the orbifold K-theory of a subgroupoid $\Lambda(X/\!\!/G)$ of $Loop^{ext}(X/\!\!/G)$ consisting of constant loops. More explicitly, $QEll^*_G(X)$ can be expressed in term of the equivariant K-theory of X and its fixed point subspaces

$$QEll^*_G(X) := \prod_{\sigma \in G^{tors}_{conj}} K^*_{\Lambda_G(\sigma)}(X^\sigma) = \Big(\prod_{\sigma \in G^{tors}} K^*_{\Lambda_G(\sigma)}(X^\sigma) \Big)^G. \tag{1}$$

We can also construct quasi-elliptic cohomology for orbifolds, which shows that orbifold quasi-elliptic cohomology can also be constructed as K-theory of a loop space. We extend the definition of $Loop^{ext}(X/\!\!/G)$ and study the bibundles from $S^1/\!\!/*$ to a Lie groupoid $\mathbb{G}$. Next we construct the orbifold $\Lambda(\mathbb{G})$ extending the definition of the constant loop subgroupoid $\Lambda(X/\!\!/G)$. The orbifold quasi-elliptic cohomology $QEll^*(\mathbb{G})$ is defined to be $K^*_{orb}(\Lambda(\mathbb{G}))$. When $\mathbb{G}$ is a global quotient $X/\!\!/G$, $QEll^*(\mathbb{G})$ is exactly the theory $QEll^*_G(X)$ defined by (1).

In [25], Ganter explains that Tate K-theory is really a cohomology theory for orbifolds based on Devoto's definition. We have the relation

$$QEll^*(\mathbb{G}) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = K^*_{Tate}(\mathbb{G}). \tag{2}$$

Therefore, via the loop construction of quasi-elliptic cohomology, we can get another loop construction of equivariant and orbifold Tate K-theory, other than that given by Ganter.

Moreover, we define an involution for the theory which is compatible with its geometric interpretation. As equivariant K-theory, quasi-elliptic cohomology also has the Real and the real version, which is shown in Chapter 5, [38].

Currently we are studying whether the loop space construction of quasi-elliptic cohomology can be applied to the construction of elliptic cohomology and thus interpret Witten's conjecture, which is explained more in Section 7.

2. Geometric structure on elliptic curves

Ever since the concept of elliptic cohomology theories was given, people have been exploring the relation between the geometric structure on elliptic curves and elliptic cohomology. In 1995 a classification of the level structures on the formal group of Morava E-theory is given in [2]. In 1998 Strickland proved in [59] that the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group. The Morava E-theory with chromatic level 2 is an elliptic cohomology. Then a conjecture was formulated that the classification of the finite subgroups of any elliptic curve can all be expressed in the same form as Strickland's theorem. In 2015 Stapleton proved in [55] this result for generalized Morava E-theory via transchromatic character theory [57] [58]. We proved that for quasi-elliptic cohomology and Tate K-theory we also have Strickland's theorem.

In all the cases above the power operation serves as a bridge connecting the homotopy theory and its formal group. Power operations in elliptic cohomology arise from isogenies of the underlying elliptic curve [3]. Moreover, Ando, Hopkins and Strickland studied the power operation for Morava E-theories in [5]. Ganter constructed the power operation for equivariant Tate K-theory in [24] and that for orbifold Tate K-theory in [25]. The questions arise whether there is a power operation of quasi-elliptic cohomology exhibiting the relation between the cohomology theory and Tate curve and what advantages it has over the power operation of Tate K-theory. The explicit relation between quasi-elliptic cohomology and equivariant K-theories guarantees the existence of a power operation analogous to the Atiyah's power operation on equivariant K-theories [7]. Ganter indicates in [24] and [25] that quasi-elliptic cohomology has a power operation reflecting the geometric nature of the Tate curve.

We construct a power operation $\{\mathbb{P}_n\}_{n\geq 0}$ for quasi-elliptic cohomology via explicit formulas that interwine the power operation in K-theory and natural symmetries of the free loop space. This is also the starting point to study the classification of the finite subgroups of the Tate curve. We have the theorem below.

THEOREM 2.1. *(Section 4.2,* [35]*) Quasi-elliptic cohomology has a power operation*

$$\mathbb{P}_n : QEll_G(X) \longrightarrow QEll_{G\wr\Sigma_n}(X^{\times n})$$

that is elliptic in the sense: $\mathbb{P}_n$ *can be uniquely extended to the stringy power operation*

$$P_n^{string} : K_{Tate}(X/\!\!/G) \longrightarrow K_{Tate}(X^{\times n}/\!\!/(G\wr\Sigma_n))$$

of the Tate K-theory in [24]*, which is elliptic in the sense of* [4].

The construction of the power operation $\{\mathbb{P}_n\}_{n\geq 0}$ mixes power operation in K-theory with natural operation of dilating and rotating loops. One advantage of it is it can be generalized to other equivariant cohomology theories whereas the stringy power operation of Tate K-theory cannot.

In addition, an elliptic power operation for orbifold quasi-elliptic cohomology exists. We constructed it in [36] and [38], which satisfies the axioms that Ganter spelled out for orbifold theories with power operations in [25]. It can uniquely extend to the power operation for orbifold Tate K-theory in [25], which is closely related to the level structure and isogenies on Tate curve.

As indicated in [5], power operation connects elliptic cohomology and elliptic curve. Via the power operation $\{\mathbb{P}_n\}_{n\geq 0}$ in Theorem 2.1, we explore the geometric nature of the Tate curve. In this process, quasi-elliptic cohomology plays a key role that reduces facts such as the classification of geometric structures on the Tate curve into questions in representation theory.

It is already illuminating to study the power operation $\{\mathbb{P}_n\}_{n\geqslant 0}$ when X is a point with trivial group action. A big ingredient then is understanding $QEll(\mathrm{pt}/\!\!/\Sigma_N)$. Applying that we prove in [35] that the Tate K-theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve, which is analogous to the principal result in Strickland [59] that the Morava $E-$theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of the formal group $\mathbb{G}_E$.

Our main conclusion in this section is Theorem 2.2.

THEOREM 2.2. *(Theorem 6.4,* [35]*) The Tate K-theory of symmetric groups modulo a certain transfer ideal,* $K_{Tate}(pt/\!\!/\Sigma_N)/I_{tr}^{Tate}$*, classifies finite subgroups of the Tate curve.*

Equivariant K-theory $K_G(\mathrm{pt})$ of a point is the representation ring of G. As shown in (1), quasi-elliptic cohomology can be expressed by equivariant K-theories. Thus, we can compute quasi-elliptic cohomology of a point by computing the representation ring of each $\Lambda_G(\sigma)$ whereas we cannot compute Tate K-theory directly with representation theory. To prove Theorem 2.2, we first compute the quasi-elliptic cohomology of symmetric groups modulo the transfer ideal, i.e. $QEll(\mathrm{pt}/\!\!/\Sigma_N)/\mathcal{I}_{tr}^{QEll}$. Then, applying the relation (2) between Tate K-theory and quasi-elliptic cohomology, we get the formula for $K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)/I_{tr}^{Tate}$. The power operation $\{\mathbb{P}_n\}_{n\geqslant 0}$ in Theorem 2.1 provides the isomorphism relating $K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)/I_{tr}^{Tate}$ with the ring that classifies the finite subgroups of the Tate curve. The role of quasi-elliptic cohomology in the proof is crucial.

Applying the same idea, we proved that the Tate K-theory of any finite abelian group A modulo a certain transfer ideal classifies the $A-$Level structures of the Tate curve. The result will appear in a coming paper [43].

Another contribution of the power operation $\{\mathbb{P}_n\}_{n\geqslant 0}$ is that we can define an operation $\{\overline{P}_N\}_{N\geq 0}$ of quasi-elliptic cohomology via it. It is a ring homomorphism and is analogous to the Adams operation of equivariant K-theories. Moreover, it uniquely extends to an additive operation of the Tate K-theory

$$\overline{P^{string}}_n : K_{Tate}(X/\!\!/G) \longrightarrow K_{Tate}(X/\!\!/G) \otimes_{\mathbb{Z}((q))} K_{Tate}(\mathrm{pt}/\!\!/\Sigma_N)/I_{tr}^{Tate}$$

constructed in [24].

Applying the Strickland's theorem in [59], Ando, Hopkins and Strickland show in [5] that the additive power operation of Morava E-theories

$$E^0 \longrightarrow E(B\Sigma_{p^k})/I_{tr}$$

has a nice algebra-geometric interpretation in terms of the formal group and it takes the quotient by the universal subgroup. The additive operation $\{\overline{P}_N\}_{N\geq 0}$ also contains subtle geometric information. Via it, we construct the universal finite subgroup of the Tate curve in [39].

3. Twisted quasi-elliptic cohomology and twisted equivariant elliptic cohomology theory

It is a classical result that the Chern character maps complex K-theory isomorphically onto complex cohomology. In the equivariant case, this is not always true. In [53] Rosu described $K_T^*(X)\otimes\mathbb{C}$ in terms of $H_T^*(X)\otimes\mathbb{C}$ via a globalised Chern character with T an abelian compact Lie group. Notably, an interpretation of the equivariant Chern character was given in [14] as a version of super holonomy on constant super loops in $X/\!\!/G$. In [23, Theorem 3.9] Freed, Hopkins and Teleman generalised the result to the twisted case. They described twisted equivariant K-theory via the twisted Chern character in terms of twisted equivariant cohomology of fixed-point sets with coefficients in certain equivariantly flat complex line bundles. Moreover, the construction of Chern character can be carried to higher chromatic level. In [26, Section 3] Ganter discussed elliptic Chern character map in the context of equivariant elliptic cohomology.

Based on the idea in [53], we construct a Chern character map from complex quasi-elliptic cohomology to Devoto's equivariant elliptic cohomology [17] when the group is finite. It is constructed from fixed point spaces. In this way we provide another Chern character map with the information of elliptic cohomology built in. The key role in the construction are the Atiyah-Segal map [8] and the Chern Character of complex K-theory.

In addition, we show the construction can be carried to twisted theories. In [13], to demonstrate the relation between physics and elliptic cohomology, Berwick-Evans constructed a twisted equivariant refinement of $TMF\otimes\mathbb{C}$ motivated by the geometry of $2|1-$dimensional supersymmetric sigma models and defined twisted equivariant elliptic cohomology. We first construct twisted quasi-elliptic cohomology $QEll^*_\alpha(-)$. It has the relation with twisted equivariant Tate K-theory $\alpha K_{Tate}(X/\!\!/G)$ [18] as below. As the relation between quasi-elliptic cohomology and Tate K-theory (2),

$$QEll^*_\alpha(X/\!\!/G)\otimes_{\mathbb{Z}[q^\pm]}\mathbb{Z}((q)) \cong \alpha K_{Tate}(X/\!\!/G)$$

As discussed in Section 1, quasi-elliptic cohomology is the orbifold K-theory of a subgroupoid of an extended category of bibundles from $S^1/\!\!/*$ to $X/\!\!/G$ consisting of constant loops. By extended we mean the rotation of circles are added as morphisms into this category of bibundles. We construct the twisted version of the loop space via bibundles. The twisted loops are principal $\mathbb{T}-$bundles over bibundles from $S^1/\!\!/*$ to $X/\!\!/G$. A subgroupoid of the category of twisted loops consisting of constant loops provides as a loop space model for twisted quasi-elliptic cohomology, thus, for twisted equivariant Tate K-theory as well.

We connect twisted quasi-elliptic cohomology with twisted equivariant elliptic cohomology theory via a twisted Chern character map. When the twist is trivial, it specialized to the untwisted Chern character map.

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4. Level Structure and Morava E-theories

Power operations for Morava E-theory have been studied for the last three decades. Due to the close connection between Morava E-theory and the arithmetic geometry of the universal deformation $\mathbb{G}$, the best strategy for understanding these operations has been to provide an algebro-geometric description of these maps when possible. In order to aid this endeavor, Strickland proved that, after taking the quotient by a certain transfer ideal, the E-cohomology of the symmetric group is the ring of functions on the scheme that represents subgroup schemes of a particular order in the universal deformation formal group. Ando, Hopkins, and Strickland proved that the power operation

$$P_m/I_{tr}\colon E^0 \to E^0(B\Sigma_m)/I_{tr}$$

is the ring of functions on the map of formal schemes that takes a deformation with a choice of subgroup of order m to the quotient deformation - a canonical deformation with formal group given by the quotient. The target of the power operations can be simplified by making use of the fact that there is an injection

$$E^0(B\Sigma_m)/I_{tr} \hookrightarrow \prod_{A\subset\Sigma_m \text{transitive}} E^0(BA)/I_{tr},$$

where $I_{tr} \subset E^0(BA)$ is another particular transfer ideal. The E^0-algebra $E^0(BA)/I_{tr}$ is very closely related to the scheme of A^*-level structures on $\mathbb{G}$. This scheme was introduced by Drinfeld in [19, Section 4] and has been much studied by arithemetic geometers. In particular, it is a complete local domain and a choice of basis for A^* gives rise to a system of parameters for the maximal ideal. These E^0-algebras are complicated but somewhat accessible.

Any kind of explicit calculation of these power operations above height 1 turns out to be quite difficult. These calculations were initiated by Rezk and have been successfully continued by Zhu. However, certain variants of this power operation are more geometric. In particular, work of Ganter and I describe and study the power operations on Tate K-theory and those of quasi-elliptic cohomology completely geometrically. And both theories are closely related to height 2 Morava E-theory. In particular, the generalized transchromatic character maps by Stapleton [57] [58] can be used to approximate Morava E-theory by multiple loopy extensions of p-adic K-theory that are essentially coarse versions of Tate K-theory. As more variants of E-theory and complex K-theory arise in nature, it is worth gaining a better understanding of the target of the power operation.

We restrict our attention to the simplest loopy extension of Morava E-theory: $E^0(\mathcal{L}^k(-))$. Although this "cohomology theory" does not seem to inherit power operations from E-theory, the loopy extensions of E-theory that are expected to have power operations map to it. Thus the first E^0-algebra that one might want to study in this context is $E^0(\mathcal{L}^k(B\Sigma_m))/I_{tr}$. This was accomplished by Schlank and Stapleton [55] who gave an algebro-geometric description in terms of subgroup schemes of order m in $\mathbb{G} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^k$ generalizing and reproving Strickland's result in [59]. As level structures are more often to work with than subgroups and also more amenable to calculation, in this paper we study $E^0(\mathcal{L}^k(BA))/I_{tr}$ for A finite abelian. We prove a variety of results analogous to the more classical results regarding $E^0(BA)/I_{tr}$ (reference?). Among other things, we study the product decomposition of $E^0(\mathcal{L}^k(BA))/I_{tr}$ by making use of certain families of subgroups of A, give an algebro-geometric description of these E^0-algebras in terms of level structures on $\mathbb{G} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^k$, and describe the relation to $E^0(\mathcal{L}^k(B\Sigma_m))/I_{tr}$.

To state the main result more precisely, we need some setup. Assume that A is a finite abelian p-group and let $f\colon \mathbb{Z}_p^k \to A$ be a continuous map of abelian groups. Let $\mathcal{F}_f$ be the minimal family of subgroups of A containing

$$\{H \subset A | H \text{ is proper and } f \text{ factors through } H\}.$$

Associated to $\mathcal{F}_f$ is the transfer ideal $I_{\mathcal{F}_f} \subseteq E^0(BA)$, which is generated by the image of the transfers along $H \in \mathcal{F}_f$.

The Pontryagin dual of $\mathbb{Z}_p^k$ is $(\mathbb{Q}_p/\mathbb{Z}_p)^k$ and the Pontryagin dual of f is

$$f^*\colon A^* \to (\mathbb{Q}_p/\mathbb{Z}_p)^k.$$

We define $\mathrm{Level}_f(A^*, \mathbb{G})$ to be the pullback of schemes

$$\begin{tikzcd} \mathrm{Level}_f(A^*,\mathbb{G}) \arrow[r] \arrow[d] & \mathrm{Level}(\ker(f^*),\mathbb{G}) \arrow[d] \\ \mathrm{Hom}(A^*,\mathbb{G}) \arrow[r, "f^*"] & \mathrm{Hom}(\ker(f^*),\mathbb{G}). \end{tikzcd} \tag{3}$$

THEOREM 4.1. *There is a canonical isomorphism of E^0-algebras*

$$(E^0(BA)/I_{\mathcal{F}_f})^{\mathrm{free}} \cong \mathcal{O}_{\mathrm{Level}_f(A^*,\mathbb{G}\oplus\mathbb{T}')},$$

where $\mathcal{O}_{\mathrm{Level}(A^*,\mathbb{G}\oplus\mathbb{T}')}$ denotes the ring of functions on $\mathrm{Level}(A^*, \mathbb{G} \oplus \mathbb{T}')$ and $(E^0(BA)/I_{\mathcal{F}_f})^{\mathrm{free}}$ denotes the torsion-free part of the ring $E^0(BA)/I_{\mathcal{F}_f}$.

This result is proved by first using Hopkins–Kuhn–Ravenel character theory to produce a canonical isomorphism

$$(E^0(\mathcal{L}^k(BA))/I_{tr})^{\mathrm{free}} \cong \mathcal{O}_{\mathrm{Level}(A^*,\mathbb{G}\oplus(\mathbb{Q}_p/\mathbb{Z}_p)^k)}$$

and then analyzing the fibers of the map $\mathrm{Level}(A^*, \mathbb{G} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^k) \to \mathrm{Hom}(A^*, (\mathbb{Q}_p/\mathbb{Z}_p)^k)$ given by post-composition with the projection $\mathbb{G} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^k \to (\mathbb{Q}_p/\mathbb{Z}_p)^k$.

The definition (3) of $\mathrm{Level}_f(A^*, \mathbb{G})$ guarantees a simple though non-canonical description of it and the existence of a decomposition of transfer ideal $I_{\mathcal{F}_f}$. If we have a decomposition $A = M \oplus K$ satisfying that $e \times K \subset \mathrm{im}\, f$ and $\mathrm{im}\, f_M \subset pM$ where $f_M\colon \mathbb{L}' \to M$ is the composite of f with the projection onto M, then we have an isomorphism

$$E^0(BA)/I_{\mathcal{F}_f} \cong E^0(BM)/I_M \otimes_{E^0} E^0(BK).$$

This indicates that $E^0(BA)/I_{\mathcal{F}_f}$ is an intermediate state between level structures and subgroups of A and f is the factor that determines the proportion of each part in it.

Moreover, some classical interpretation of $E^0(BA)/I_A$ can be generalized to $E^0(BA)/I_{\mathcal{F}_f}$. Define $S \subset E^0(\mathcal{L}^h BA)$ to be the pulled back Euler classes of nontrivial irreducible representations of the quotients of A. And define S_f denote the subset of S consisting of representations of A/H with $H \in \mathcal{F}_f$. Motivated by the idea in [34, Proposition 6.5], we show the localization map $E^0(BA) \to S_f^{-1}E^0(BA)$ factors through the quotient $E^0(BA)/I_{\mathcal{F}_f}$ and thus gives a canonical map of E^0-algebras

$$E^0(BA)/I_{\mathcal{F}_f} \to S_f^{-1}E^0(BA).$$

In addition, based on [31, Proposition 3.20], we show that the zeroth homotopy group of the geometric fixed points for the family $\mathcal{F}_f$ of the Borel completion $\underline{E}$ of the spectrum E is isomorphic to $S_f^{-1}E^0(BA)$.

We compare $\mathrm{Level}_f(A^*, \mathbb{G})$ with the subgroup scheme $\mathrm{Sub}_{p^k}^{\mathrm{im}\, f}(\mathbb{G} \oplus \mathbb{T}')$ defined in [55]. There is a canonical map

$$\mathrm{im}\colon \mathrm{Level}_f(A^*, \mathbb{G} \oplus \mathbb{T}') \to \mathrm{Sub}_{p^k}^{\mathrm{im}\, f^*}(\mathbb{G} \oplus \mathbb{T}')$$

by sending a level structure $l\colon A^* \hookrightarrow \mathbb{G} \oplus \mathbb{T}'$ such that the composite $A^* \hookrightarrow \mathbb{G} \oplus \mathbb{T}' \to \mathbb{T}'$ is equal to f^* to the subgroup scheme $\mathrm{im}\, l$, a subgroup of $\mathbb{G} \oplus \mathbb{T}'$ that projects onto $\mathrm{im}\, f^* \subset \mathbb{T}'$. And we have the commutative diagram of formal schemes

$$\begin{tikzcd}
\mathrm{Spf}(E^0(BA)/I_{\mathcal{F}_f}) \arrow[r] \arrow[d] & \mathrm{Spf}(E^0(B\,\mathrm{im}\, if \wr \Sigma_{p^j})/I_{tr}^{[if]}) \arrow[d] \\
\mathrm{Level}_f(A^*, \mathbb{G}\oplus\mathbb{T}') \arrow[r] & \mathrm{Sub}_{p^k}^{\mathrm{im}\, f}(\mathbb{G}\oplus\mathbb{T}').
\end{tikzcd}$$

where the right vertical map is that in [55, Proposition 7.12].

5. Equivariant orthogonal spectra of quasi-elliptic cohomology

Equivariant homotopy theory is homotopy theory of topological $G-$spaces. Mandell, May, Schewede and Shipley built several good model categories of equivariant spectra [51]. Equivariant orthogonal spectra, as shown in [50], is one of them. An orthogonal $G-$spectrum is defined from a $\mathcal{I}_G-$functor with $\mathcal{I}_G$ the category of orthogonal $G-$representations.

Ginzburg, Kapranov and Vasserot have the conjecture [30] that any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A. In his thesis [28], Gepner presented a construction of the equivariant elliptic cohomology that satisfies a derived version of the Ginzburg-Kapranov-Vasserot axioms. We are interested in answering this question from a different perspective and trying to give an explicit construction of orthogonal $G-$spectrum for each elliptic cohomology theory.

One advantage of quasi-elliptic cohomology is that it is built using equivariant topological K-theory, each aspect of which has been studied thoroughly. Some constructions on quasi-elliptic cohomology can be made simpler than most elliptic cohomology theories, including the Tate K-theory. So we answer the question below first.

QUESTION 5.1. *Is there an orthogonal $G-$spectrum representing $QEll_G^*(-)$?*

Applying equivariant homotopy theory, we construct an orthogonal $G-$spectrum representing quasi-elliptic cohomology [37].

THEOREM 5.2. ([37]) *For each compact Lie group G, there exists a commutative I_G-FSP $(QE(G,-), \eta^{QE}, \mu^{QE})$ representing $QEll_G^*$.*

The construction of $(QE(G,-), \eta^{QE}, \mu^{QE})$ is explicit. The idea and technique can be applied to a family of theories, including generalized Morava E-theories and equivariant Tate K-theory. Moreover, we show in [37] that some signature properties of equivariant theories $\{E_G^*\}_G$ can be inherited by the theory $E^*(\Lambda(-))$ after composing E with the loop space functor $\Lambda(-)$ defined in Section 1, including the properties below.

- The theories $\{E_G^*\}_G$ have the change-of-group isomorphism.
- The theories $\{E_G^*\}_G$ are H_∞, i.e. they have power operations.
- There exists a I_G-FSP representing $E_G^*(-)$.

6. Globalization of quasi-elliptic cohomology

At the early beginning of equivariant homotopy theory people noticed that certain theories, equivariant stable homotopy theory, equivariant bordism, equivariant K-theory, etc, naturally exist not only for one particular group but for all groups in a specific class. This observation motivated the birth of global homotopy theory. The idea of global orthogonal spectra was first inspired in Greenlees and May [32]. In [56] the concept of orthogonal spectra is introduced, which is defined from $\mathbb{L}-$functors with $\mathbb{L}$ the category of inner product real spaces. Each global spectrum consists of compatible $G-$spectra with G across the entire category of groups and they reflect any symmetry. Globalness is a measure of the naturalness of a cohomology theory. Several models of global homotopy theories have been established, including that by Bohmann [16] and Schwede [56]. The two model categories of global spectra are Quillen equivalent, as shown in [16]. But it is unclear whether global elliptic cohomology theory exists and how we should construct it.

In Remark 4.1.6 [56], Schwede discussed the relation between orthogonal $G-$spectra and global spectra. We have the question associated to the underlying orthogonal $G-$spectrum of the I_G-FSP $(QE(G,-), \eta^{QE}, \mu^{QE})$ in Theorem 5.2.

QUESTION 6.1. *Can $\{(QE(G,-), \eta^{QE}, \mu^{QE})\}_G$ arise from an orthogonal spectrum?*

Ganter showed that $\{QEll^*_G\}_G$ have the change-of-group isomorphism, which is a good sign that quasi-elliptic cohomology may be globalized. By the discussion in Remark 4.1.6 [56], however, the answer to QUESTION 6.1 is no. Then it is even more difficult to see whether each elliptic cohomology theory, whose form is more intricate and mysterious than quasi-elliptic cohomology, can be globalized in the current setting.

Our solution is to establish a new global homotopy theory where quasi-elliptic chomomology can fit into. We hope that it is easier to judge whether a cohomology theory, especially an elliptic cohomology theory, can be globalized in the new theory. In addition we want to show that the new global homotopy theory is equivalent to the current global homotopy theory.

We construct in [38] a category D_0 to replace $\mathbb{L}$ whose objects are (G, V, ρ) with V an inner product vector space, G a compact group and ρ a faithful group representations

$$\rho : G \longrightarrow O(V),$$

and whose morphism $\phi = (\phi_1, \phi_2) : (G, V, \rho) \longrightarrow (H, W, \tau)$ consists of a linear isometric embedding $\phi_2 : V \longrightarrow W$ and a group homomorphism $\phi_1 : \tau^{-1}(O(\phi_2(V))) \longrightarrow G$, which makes the diagram (4) commute.

$$\begin{tikzcd} G \arrow[r, "\rho"] & O(V) \arrow[d, "\phi_{2*}"] \\ \tau^{-1}(O(\phi_2(V))) \arrow[u, "\phi_1"] \arrow[r, "\tau"] & O(W) \end{tikzcd} \tag{4}$$

In other words, the group action of H on $\phi_2(V)$ is induced from that of G. Intuitively, the category D_0 is obtained by adding the restriction maps between representations into the category $\mathbb{L}$.

Instead of the category of orthogonal spaces, we study the category of D_0-spaces. The category of orthogonal spaces is a full subcategory of the category D_0T of D_0-spaces. Apply the idea of diagram spectra in [51], we can also define D_0-spectra and D_0-FSP.

Moreover, we notice that if the equivariant homotopy theories $\{E^*_G(-)\}_G$ is represented by a D_0-spectrum X, it has the property that $X(G, V) \simeq_H X(H, V)$ for any closed subgroup H of G. So what we really need to study are D^W_0-spectra.

DEFINITION 6.2 (The category $Sp^{D_0}_W$)**.** *A D^W_0-spectrum X is both a D_0-spectrum and a D_0-space that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an $H-$weak equivalence. The category $Sp^{D_0}_W$ is the category of D^W_0-spectra.*

Combining the orthogonal $G-$spectra of quasi-elliptic cohomology together, we get a well-defined unitary D^W_0-spectrum and unitary D^W_0-FSP. Thus, we can define global quasi-elliptic cohomology in the category of D^W_0-spectra.

THEOREM 6.3. *(Theorem 7.2.3 [38], [42])There is a unitary D^W_0-FSP representing quasi-elliptic cohomology.*

Equipping a homotopy theory with a model structure is like interpreting the world via philosophy. Model category theory is an essential basis and tool to judge whether two homotopy theories describe the same world. We build several model structures on D_0T. First by the theory in [51], there is a level model structure on D_0T.

THEOREM 6.4. *(Theorem 6.3.4 [38], [42]) $Sp_W^{D_0}$ is a compactly generated topological model category with respect to the level equivalences, level fibrations and $q-$cofibrations. It is right proper and left proper.*

In addition, this new global theory describes the same world of homotopy theories as that by Schwede.

THEOREM 6.5. *([42])There is a global model structure on $Sp_W^{D_0}$ Quillen equivalent to the global model structure on the orthogonal spectra constructed by Schwede in [56].*

Moreover, in [42] we show there is one more property of $\{E^*(-)\}_G$ that can be inherited by its composition with $\Lambda(-)$, $E^*(\Lambda(-))$.

THEOREM 6.6. *If $\{E^*(-)\}_G$ can be globalized, so are $E^*(\Lambda(-))$.*

7. Research Plan

7.1. Research Plan in Bonn.

- **Projects with Matthew Young**
  1. **The Definition of $n-$vector bundles** Most prominent cohomology theories have geometric interpretations. For example, via (1-)vector bundles, we construct topological K-theory whose chromatic level is 1; via 2-vector bundles, we can construct a form of elliptic cohomology theory [9] whose chromatic level is 2. Currently, people are looking for the right definition of equivariant 2-vector bundles that can interpret a form of equivariant elliptic cohomology theory.
     One question we are interested in is whether we can generalize these concepts and define $n-$vector bundle in the reasonable way from which we can construct chromatic level n cohomology theory. Moreover, can we generalize this idea and even construct the equivariant versions of each one? This should be a long-term brilliant project. In addition, we expect there is a right version of $n-$gerbes that develop the idea of gerbes [49] and can give a loop based definition of the level n cohomology theory.
  2. **The relation between $n-$vector bundles and loop space** We will start studying this problem by exploring the relation between Tate K-theory and the K-theory of 2-vector bundles. I also discussed this problem with Professor Fei Han at the National University of Singapore before. During Young's visit in Wuhan, December 2019, we constructed quasi-elliptic cohomology for $2-$groups.
     And we expect to interpret the relation between K-theory of $n-$vector bundles and the theory $K^*_{orb}(\Lambda^n(-))$ where $\Lambda(-)$ is the loop space in the construction of the quasi-elliptic cohomology $K^*_{orb}(\Lambda(-))$.
  3. **Quasi-elliptic cohomology and Hopkins-Kuhn-Ravenel character theory.** In 1960s Atiyah and Segal showed that the map $R(G) \longrightarrow K^0(BG)$ is an isomorphism after completing $R(G)$ with respect to the ideal of virtual bundles of dimension 0. In [34] Hopkins, Kuhn and Ravenel generalized Atiyah and Segal's completion theorem. They constructed an isomorphism relating equivariant Morava E_n-theory with an equivariant cohomology theory of chromatic level 0 of a fixed point space $Fix_n(X)$. Devoto's equivariant K-theory has a Hopkins-Kuhn-Ravenel character theory [34]. Ganter expected the HKR theory for orbifold Tate K-theory to be established via Stapleton's framework of tanschromatic character map. Stapleton constructed in his paper [57] and [58], for each finite $G-$CW complex X and each positive integer n, a topological groupoid $Twist_n(X)$, whose construction is analogous to that of the groupoid $\Lambda(X/\!\!/G)$. With the topological groupoid $Fix_n(X)$ in [34], he constructed in [57] extensions of the generalized character map of Hopkins, Kuhn, and Ravenel [34] for Morava E-theory to every height between 0 and n. And with $Twist_n(X)$, in [58] he constructed the twisted

character map which can canonically recover the transchromatic generalized character map in [57]. Apply this transchromatic character theory, he and Schlank provided a new proof of Strickland's theorem in [59] that the Morava E-theory of the symmetric group has an algebro-geometric interpretation after taking the quotient by a certain transfer ideal. Moreover, he showed in [11] this extended character theory can be used to compute the total power operation for the Morava E-theory of any finite group, up to torsion.

(a) I will construct the Hopkins-Kuhn-Ravenel (HKR) character theory for quasi-elliptic cohomology and that for Tate K-theory.
(b) An unusual phenomenon that we observe is that the chromatic level of quasi-elliptic cohomology is 1 whereas that of elliptic cohomology theories is 2. Though quasi-elliptic cohomology is very close to elliptic cohomology theories, their chromatic levels are different.

We expect Gronowski's elliptic cohomology theories and quasi-elliptic cohomology theory can be related by Stapleton's generalized character map.

(4) **Hopkins-Kuhn-Ravenel Character Theory for $n-$group** Matthew Young suggests the Hopkins-Kuhn-Ravenel character theory for $n-$group can be constructed analogously when the space X is a single point space. A natural but harder question is whether we can have Hopkins-Kuhn-Ravenel character theory generally. We will start with the case $n = 2$.
(5) **Twisted Power Operation** Matthew Spong and I constructed twsited quasi-elliptic cohomology. As shown in the next section, we are curious whether this twisted theory has power operation. But before that we need to formulate the definition of twisted equivariant power operation generally. We gave a reasonable definition after studying and comparing different definitions of twisting itself, that in [23] and [6], etc. Next, we will first construct a twisted version of Atiyah's power operation for twisted equivariant K-theory and then generalize the construction to twisted quasi-elliptic cohomology.
Moreover, there has already been work on twisted Morava E-theory [54]. We will first try to see whether there is better construction of twisted Morava E-theory and then start constructing power operation for it.
(6) **Chern character map for Real twisted quasi-elliptic cohomology** In my PhD thesis [38] I constructed the Real and real version of quasi-elliptic cohomology. Matthew Young and I will construct Chern character map for the Real theory. Furthermore, we will construct Real twisted quasi-elliptic cohomology theory and the Chern character map for it. In this process we will also construct a Real version of the twisted equivariant elliptic cohomology [13].

- **Projects with Nathaniel Stapleton**
  We will construct the power operation of generalized Morava E-theory and develop further research projects on Morava E-theories.
- **Global Homotopy Theory** As shown in Section 6, I did some work on global homotopy theory in [42]. I hope to talk with Professor Stefan Schwede and improve the work.

7.2. Other Research Problems.

- **Projects with Matthew Spong**
  We have a series of projects applying the Chern character maps in Section 3.

(1) In Theorem 2.1 I show an "elliptic" power operation [35] of quasi-elliptic cohomology. Via the Chern character map, we will induce a power operation for Devoto's equivariant elliptic cohomology and compare it with the power operation in [12].
(2) Moreover, we will construct the twisted version of the elliptic power operation of quasi-elliptic cohomology. With the twisted Chern character map, we expect to construct a power operation of the twisted equivariant elliptic cohomology.
(3) In addition, as shown in [13], chromatic height 2 phenomena in the sense of [34] on twisted equivariant elliptic cohomology is discussed, which suggests a deeper connection

between physics and elliptic cohomology. We are curious whether there is a Hopkins-Kuhn-Ravenel character theory on quasi-elliptic cohomology, which is expected to be related to the character theory on twisted equivariant elliptic cohomology theory via the Chern character map.

(4) We will also study whether quasi-elliptic cohomology is a receptacle of Witten genus and thus is related to a certain two-dimensional sigma model.

(5) Defined by Nitu Kitchloo, dominant K-theory [45] is a version of equivariant K-theory of Kac-Moody groups and coincides with the usual equivariant K-theory when the groups are compact Lie groups. We plan to study under what restrictions there is a Chern character from dominant K-theory and what is the relation between it and the twisted Chern character from twisted equivariant K-theory by Freed, Hopkins and Teleman [23].

- **Projects in Georg-August-Universität Göttingen**

  During February 2020-February 2021, I will visit Georg-August-Universität Göttingen. During the visit, I will work on these families of research problems.

  (1) **Extend quasi-elliptic cohomology to the homotopy theory of stacks** In [47] Lerman explained how to embed the category of orbifolds into that of stacks. In [29] Gepner established the homotopy theory of topological groupoids and that of topological stacks and compared the two homotopy theories. These mathematical insights imply we may extend the definition of quasi-elliptic cohomology to higher K-theories. It may be of larger interest to topologists and geometrists comparing with the current definition. The study may also benefit mathematicians working on number theory and ring theory. Professor Chenchang Zhu has been working on the geometric aspect of K-theory and higher structures in differential geometry for years. We will first formulate a reasonable definition of quasi-elliptic cohomology of stacks and then apply it to solve problems in geometry.

  (2) **Explore the relation between quasi-elliptic cohomology and physics.** Mathematical physics in Göttingen is very strong, which will benefits me a lot in studying this research problem.
     - First I plan to express quasi-elliptic cohomology by group schemes since algebraic geometry is related to physics more closely than algebraic topology is.
     - Then I will construct some physical invariants associated to elliptic cohomology theories (Witten genus etc) on quasi-elliptic cohomology. I am curious what distinct properties and advantages they may have. This is also part of my projects with Matthew Spong.

  (3) **Another representing object of quasi-elliptic cohomology** Other than equivariant spectrum, Professor Chenchang Zhu suggested that another type of representing object for quasi-elliptic cohomology is two-vector bundle [9][10], which is constructed geometrically for elliptic cohomology. Two-representation and two vector-bundle are related to categorical representation theory constructed by Ganter [27]. Categorical representation theory has already shown great power in solving geometric problems. Exploring the geometric nature of quasi-elliptic cohomology and its relation with categorical representation theory, we may get valuable research outcome.
     I will construct a geometric representing object via two-vector bundles and discuss with both Professor Zhu and Dr. Ganter working on an even larger picture.

  (4) **Develop bivariant quasi-elliptic cohomology** Most homology invariants in algebraic topology do not carry over to noncommutative spaces. The main invariant in that setting is K-theory. In noncommutative geometry, K-theory can be extended to a bivariant theory, which provides powerful tools for computation of K-theory and a rich categorical structure. Quasi-elliptic cohomology is defined from $\Lambda_G(g)-$equivariant K-theories with $\Lambda_G(g)$ defined as the quotient of $C_G(g)\times\mathbb{R}$ by a cyclic group. We may replace the groups $\Lambda_G(g)$ with bigroups or cross modules in noncommutative geometry and study their action on non-Hausdorff spaces. The contribution of this replacement to the symmetries in noncommutative geometry is worth exploration. In addition, this

replacement together with bivariant K-theory can lead to an extension of quasi-elliptic cohomology in noncommutative geometry, which may bring rich extra structures.
Professor Ralf Meyer works on the structure in bivariant K-theory and applied it to study the classification of C^*-algebras. He is also an expert on the symmetries in noncommutative geometry. I plan to communicate and work with him on this.

- **Generalize the conclusions/constructions on quasi-elliptic cohomology to elliptic cohomology theories.**
  (1) One problem we work on is related to groupoids and loop spaces. In Section 1 we used a loop space constructed from bibundles in the definition of quasi-elliptic cohomology. We want to know whether the relation between any elliptic cohomology theory and loop space can be interpreted by bibundles as well.
  (2) Via quasi-elliptic cohomology theory, we get the classification theorem of finite subgroups and that of level structures on Tate curve. The form of the conclusion is the same as those on Morava E-theories, and also those on generalized Morava E-theories [55] [43]. We also want to show that the classification theorems on each elliptic curve can be expressed by the associated elliptic cohomology in the same way as that on Tate curve.

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