Almost Global Homotopy Theory

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Plan

- Preliminary: classical homotopy theory; stable homotopy theory.
- Model category;
- Equivariant homotopy theory;
- Global homotopy theory;
- Almost global homotopy theory;
- Examples: Quasi-theories.

Preliminary: classical homotopy theory

Top The category of topological spaces and continuous maps.
Top_* The category of based spaces and based maps.

a homotopy h between continuous maps f and g: $X \longrightarrow Y$

a map $h : X \times [0,1] \longrightarrow Y$ such that $f(x) = h(x,0)$ and $g(x) = h(x,1)$.
f and g are **homotopic** $f \overset{h}{\simeq} g$ if there exists a homotopy between them.
Being homotopic is an equivalence relation.

$f : X \longrightarrow Y$ is a homotopy equivalence

there exists a map $g : Y \longrightarrow X$ and $g \circ f \backsimeq id_X$ and $f \circ g \backsimeq id_Y$.
$X \backsimeq Y$. X and Y are homotopy equivalent.

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

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Homotopy groups: $\pi_n(X, x) := [S^n, X]_*$.

Example

$\pi_0(S^0) = S^0$. $\pi_1(S^1) \cong \mathbb{Z}$. $\pi_n(S^n) \cong \mathbb{Z}$.

$f : X \longrightarrow Y$ is a weak homotopy equivalence

$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

- Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.
- Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

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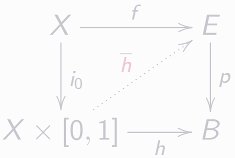
Fibration

Homotopy Lifting Property

$p : E \longrightarrow B$ is called a fibration if $p : E \longrightarrow B$ satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.



Example

• constant map; • covering space; • fiber bundle.

Serre fibration

A $p : E \longrightarrow B$ satisfying the homotopy lifting property with respect to $X = D^n$.

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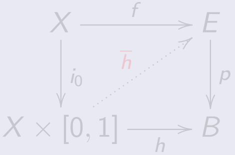

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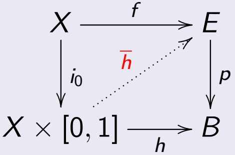

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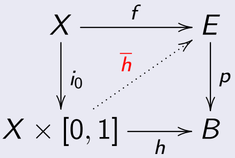

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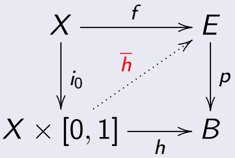

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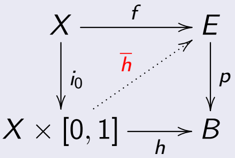

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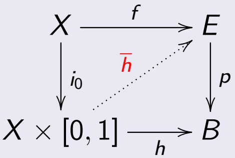

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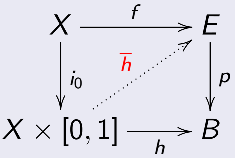

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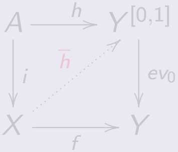
Cofibrations: dual to fibration

Homotopy Extension Property

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.



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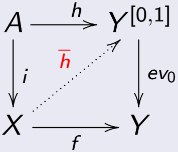

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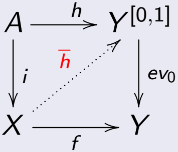

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Spectrum

a sequence $\{X_n, \sigma_n\}_n$ where X_n are based spaces and $\sigma_n : S^1 \wedge X_n \longrightarrow X_{n+1}$ are based maps.

Stable homotopy groups

$$\pi_n^S(X) = colim_{l \to \infty} \pi_{n+l}(S^l \wedge X).$$

Spectrum defines homology and cohomology

$$H_n(Y) = colim_{l \to \infty} \pi_{n+l}(Y_+ \wedge X_l);$$

$$H^n(Y) = colim_{l \to \infty} [S^l \wedge Y_+, X_{n+l}].$$

Brown Representation Theorem

(i) For each homology theory, there exists a spectrum representing it.
(ii) For each cohomology theory, there exists a spectrum representing it.

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Spectrum defines homology and cohomology

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What is "homotopy theory"?

Motivating example: the category of topological spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Features

- **2-out-of-3**: If two of f, g, gf are weak homotopy equivalences, then so is the third.
- **Retracts**:

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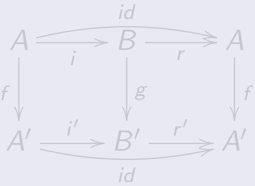

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Quillen's perspective: Model Category

Definition: A model structure on a category $\mathcal{C}$

• Weak Equivalence • Fibration • Cofibration.

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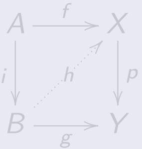

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Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
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$G-$CW complex

X^0: disjoint union of orbits G/H.
X^{n+1}: attach $G-$cells $G/H \times D^{n+1}$ to X^n along attaching $G-$maps

$$G/H \times S^n \longrightarrow X^n.$$

Equivariant homotopy group

$$GTop \longrightarrow [Orb_G^{op}, Top]$$
$$X \mapsto (G/H \mapsto X^H)$$

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Elmendorf's theorem

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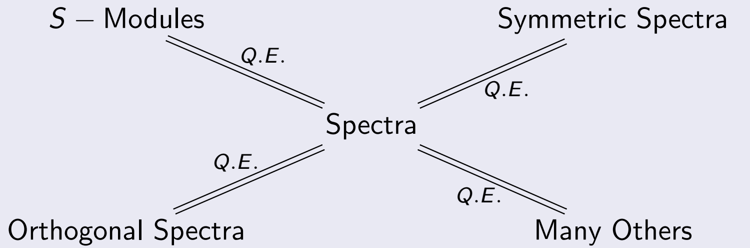
Right Philosophy for Stable Homotopy Theory (1990s) [MMSS]



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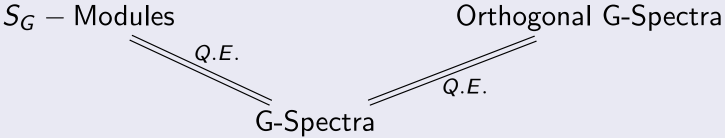

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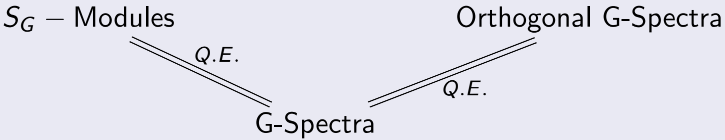

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Orthogonal G−spectra.

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Combine the best feature of other models.

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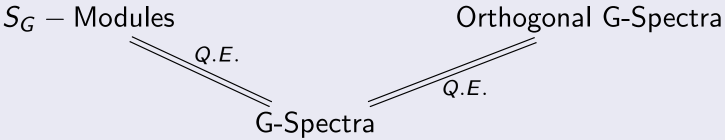

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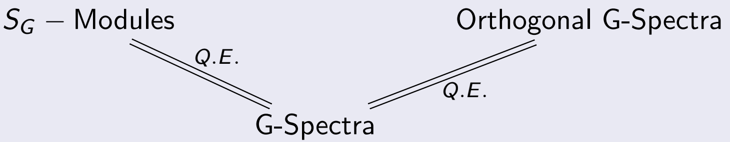

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A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Equivariant notion of a functor with smash product

An $\mathcal{I}_G-$FSP is an $\mathcal{I}_G-$space X with

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The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG. K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle.$ $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
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Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
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It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

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The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An **orthogonal space** is a continuous functor from $\mathbb{L}$ to the category of topological spaces.

The category $\mathbb{O}$

- objects: inner product real spaces;
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$$\xi(V, W) := \{(w, \phi) \in W \times L(V, W) | w \perp \phi(V)\}$$

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An **orthogonal spectrum** is a based continuous functor from $\mathbb{O}$ to the category of based topological spaces.

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- level model structure;
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- global model structure;

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$$\pi_0^G(X) = \text{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

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A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f) : \pi_k^G(X) \xrightarrow{\cong} \pi_k^G(Y)$ for all compact Lie groups G and all integers k.

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- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
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Observation: restriction maps don't need to be identity maps.

- $\{E^n_G, \sigma_{G,n}\}_{n,G}$: equivariant spectra representing $\{E^*_G(-)\}_G$.
  $E^n_G \simeq_H E^n_H$ for $H \overset{i}{\hookrightarrow} G$.
- For an orthogonal spectrum X, $X(i^*(V)) = i^*X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
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$$\begin{array}{ccc} G & \longrightarrow & O(V) \\ \uparrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2^*} \\ H \cap O(V) & \longrightarrow & O(W) \end{array}$$

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The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
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The right subject: D_0^W-spectra [Huan]

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A D_0^W-spectrum X is both a D_0-spectrum and a D_0-space in D_0T^W.

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$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

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Divisible group: a better algebraic object associated to an elliptic curve than formal group.

the generalized Tate K-theory and generalized quasi-elliptic cohomology

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_n \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}_p^n} \mathbb{Q}_p^n)[p^k]) = K^0_{n,Tate}(B\mathbb{Z}_{p^k})$.
The corresponding quasi-theory:
$QK^*_{n,G}(X) := K^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma\in G^n_z} K^*_{\Lambda^n_G(\sigma)}(X^\sigma)$.

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Theorem

If the theory $\{E^*_{n,G}(-)\}_G$ can be globalized, there is a D_0^W-spectrum representing the quasi-theory $\{QE^*_{n,G}(-)\}_G$.
In particular, quasi-elliptic cohomology, the quasi-theory of Tate K-theory, can be globalized in almost global homotopy theory.

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My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

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Thank you.

Some references

https://huanzhen84.github.io/zhenhuan/Huan-HUST-2018.pdf

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