Almost Global Homotopy Theory

Zhen Huan

Sun Yat-sen University

October 10, 2018

Plan

- Preliminary: classical homotopy theory.
- Model category;
- Equivariant homotopy theory;
- Global homotopy theory;
- Almost global homotopy theory;
- Examples: Quasi-theories.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f : X \longrightarrow Y$ is a weak homotopy equivalence if
$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

• Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.

• Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f: X \longrightarrow Y$ is a weak homotopy equivalence if
$f_*: \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

• Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.

• Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f: X \longrightarrow Y$ is a weak homotopy equivalence if
$f_*: \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

• Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.

• Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f : X \longrightarrow Y$ is a weak homotopy equivalence if
$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

• Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.

• Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f : X \longrightarrow Y$ is a weak homotopy equivalence if
$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

- Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.
- Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f : X \longrightarrow Y$ is a weak homotopy equivalence if
$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

- Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.
- Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f : X \longrightarrow Y$ is a weak homotopy equivalence if
$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

• Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.

• Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

What is "homotopy theory"?

Motivating example: the category of topological spaces:

- What are its key features?
- What are its key components?

weak homotopy equivalence

$f : X \longrightarrow Y$ is a weak homotopy equivalence if
$f_* : \pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for each n and each x.

homotopy equivalence $\Rightarrow$ weak homotopy equivalence.

CW-complex: nice spaces

X^0: a discrete set.
X^{n+1}: attach $(n+1)-$cells D^{n+1} to X^n along attaching maps $S^n \longrightarrow X^n$.

• Any Hausdorff topological space is weak homotopy equivalent to a CW-complex.

• Weak homotopy equivalences between connected CW-complexes are homotopy equivalences.

Fibration

$p : E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ \downarrow{\scriptstyle i_0} & \overset{\overline{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[h]{} & B \end{array}$$

Example

- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p : E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

Fibration

$p : E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ \Big\downarrow{\scriptstyle i_0} & \overset{\overline{h}}{\nearrow} & \Big\downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[h]{} & B \end{array}$$

Example

- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p : E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

Fibration

$p : E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ \downarrow{\scriptstyle i_0} & \overset{\overline{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[h]{} & B \end{array}$$

Example

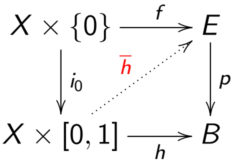
- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p : E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

Fibration

$p : E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.



Example

- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p : E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

Fibration

$p : E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ \downarrow{\scriptstyle i_0} & \overset{\overline{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[h]{} & B \end{array}$$

Example

- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p : E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

Fibration

$p : E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{\ f\ } & E \\
{\scriptstyle i_0}\downarrow & \overset{\overline{h}}{\nearrow} & \downarrow{\scriptstyle p} \\
X \times [0,1] & \xrightarrow[\ h\ ]{} & B
\end{array}
$$

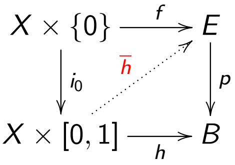
Example

- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p : E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

$p: E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f: X \longrightarrow E$ and homotopy $h: X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h}: X \times [0,1] \longrightarrow E$ making the diagram commute.



Example

- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p: E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

Fibration

$p : E \longrightarrow B$ is called a fibration if it satisfies the Homotopy Lifting Property, i.e. given any map $f : X \longrightarrow E$ and homotopy $h : X \times [0,1] \longrightarrow B$ with $h_0 = p \circ f$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow E$ making the diagram commute.

$$\begin{array}{ccc} X \times \{0\} & \xrightarrow{f} & E \\ {\scriptstyle i_0}\downarrow & \overset{\overline{h}}{\nearrow} & \downarrow{\scriptstyle p} \\ X \times [0,1] & \xrightarrow[h]{} & B \end{array}$$

Example

- constant map; covering space; fiber bundle.
- composition; pullback; product; retract; sequential inverse limits.

Serre fibration

$p : E \longrightarrow B$ satisfying the Homotopy Lifting Property w.r.t $X = D^n$.

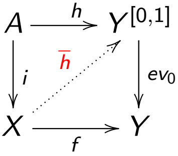
$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.

Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

Cofibrations: dual to fibration

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.

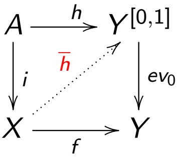


Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

Cofibrations: dual to fibration

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.

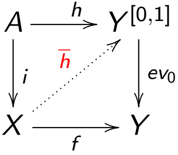


Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

Cofibrations: dual to fibration

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.

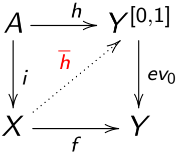


Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

Cofibrations: dual to fibration

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.

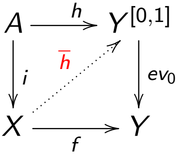


Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

Cofibrations: dual to fibration

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.

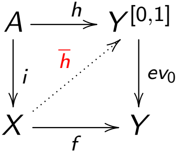


Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

Cofibrations: dual to fibration

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.

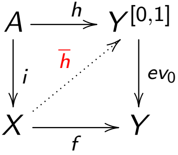


Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

Cofibrations: dual to fibration

$i : A \longrightarrow X$ is called a cofibration if $i : A \longrightarrow X$ satisfies the Homotopy Extension Property, i.e. if given any map $f : A \longrightarrow X$, homotopy $h : A \times [0,1] \longrightarrow Y$ with $h_0 = f \circ i$, there exists an extension $\overline{h} : X \times [0,1] \longrightarrow Y$ making the diagram commute.



Example

- The inclusions $\varnothing \hookrightarrow X$; $\{0\} \hookrightarrow [0,1]$; $S^{n-1} \hookrightarrow D^n$;
- composition; pushouts; coproducts; retracts; sequential colimits;
- Inclusion of a subcomplex into a CW-complex;
- Retract of a relative cell complex.

- **Weak Equivalence • Fibration • Cofibration.**
  - **2-out-of-3**: If two of f, g, gf are weak equivalences, so is the third.
  - **Retracts**:

f is a retract of g.

If g is a weak equivalence/fibration/cofibration, then so is f.

  - **Lifting**:

The trivial cofibrations have the left lifting property w.r.t. fibrations; cofibrations have the left lifting property w.r.t. trivial fibrations.

  - **Factorization**: (α, β), (γ, δ): Map($\mathcal{C}$) $\longrightarrow$Map($\mathcal{C}$).
$f = \beta(f) \circ \alpha(f)$; $f = \delta(f) \circ \gamma(f)$. $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, $\delta(f)$ is a fibration.

- Weak Equivalence • Fibration • Cofibration.
  - **2-out-of-3**: If two of f, g, gf are weak equivalences, so is the third.
  - **Retracts**:

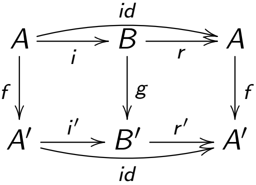
    f is a retract of g.

    If g is a weak equivalence/fibration/cofibration, then so is f.
  - **Lifting**:

    The trivial cofibrations have the left lifting property w.r.t. fibrations; cofibrations have the left lifting property w.r.t. trivial fibrations.
  - **Factorization**: (α, β), (γ, δ): Map($\mathcal{C}$) $\longrightarrow$Map($\mathcal{C}$).

    $f = \beta(f) \circ \alpha(f)$; $f = \delta(f) \circ \gamma(f)$. $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, $\delta(f)$ is a fibration.

- Weak Equivalence • Fibration • Cofibration.
  - **2-out-of-3**: If two of f, g, gf are weak equivalences, so is the third.
  - **Retracts**:



f is a retract of g.

If g is a weak equivalence/fibration/cofibration, then so is f.

  - **Lifting**:

The trivial cofibrations have the left lifting property w.r.t. fibrations; cofibrations have the left lifting property w.r.t. trivial fibrations.

  - **Factorization**: (α, β), (γ, δ): $\mathrm{Map}(\mathcal{C}) \longrightarrow \mathrm{Map}(\mathcal{C})$.

$f = \beta(f) \circ \alpha(f)$; $f = \delta(f) \circ \gamma(f)$. $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, $\delta(f)$ is a fibration.

- Weak Equivalence • Fibration • Cofibration.
  - **2-out-of-3**: If two of f, g, gf are weak equivalences, so is the third.
  - **Retracts**:

$$\begin{array}{ccccc} A & \xrightarrow{i} & B & \xrightarrow{r} & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ A' & \xrightarrow{i'} & B' & \xrightarrow{r'} & A' \end{array} \qquad r\circ i = id,\ r'\circ i' = id$$

f is a retract of g.

If g is a weak equivalence/fibration/cofibration, then so is f.

  - **Lifting**:

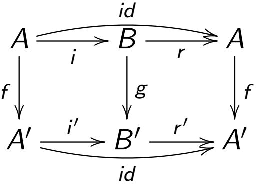
$$\begin{array}{ccc} A & \xrightarrow{f} & X \\ i\downarrow & \overset{h}{\nearrow} & \downarrow p \\ B & \xrightarrow{g} & Y \end{array}$$

The trivial cofibrations have the left lifting property w.r.t. fibrations; cofibrations have the left lifting property w.r.t. trivial fibrations.

  - **Factorization**: (α, β), (γ, δ): Map($\mathcal{C}$) $\longrightarrow$Map($\mathcal{C}$).
$f = \beta(f) \circ \alpha(f)$; $f = \delta(f) \circ \gamma(f)$. $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, $\delta(f)$ is a fibration.

Quillen's perspective: Model Category

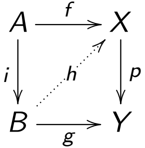
- Weak Equivalence • Fibration • Cofibration.
  - **2-out-of-3**: If two of f, g, gf are weak equivalences, so is the third.
  - **Retracts**:



f is a retract of g.

If g is a weak equivalence/fibration/cofibration, then so is f.

  - **Lifting**:



The trivial cofibrations have the left lifting property w.r.t. fibrations; cofibrations have the left lifting property w.r.t. trivial fibrations.

  - **Factorization**: (α, β), (γ, δ): $\mathrm{Map}(\mathcal{C}) \longrightarrow \mathrm{Map}(\mathcal{C})$.
$f = \beta(f) \circ \alpha(f)$; $f = \delta(f) \circ \gamma(f)$. $\alpha(f)$ is a cofibration, $\beta(f)$ is a trivial fibration, $\gamma(f)$ is a trivial cofibration, $\delta(f)$ is a fibration.

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

$\mathbf{Ho(Top)}$: the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Structures on Topological Spaces

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Structures on Topological Spaces

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Structures on Topological Spaces

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Structures on Topological Spaces

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Structures on Topological Spaces

Classical Quillen Model Structure on Topological Spaces

- **weak homotopy equivalence**
- **Serre fibration**
- **retract of relative cell complex**

Hurewicz Model Structure on Topological Spaces

- **homotopy equivalence**
- **fibration**
- **closed cofibration**

Ho(Top): the associated homotopy category

- objects: topological spaces.
- morphisms: homotopy classes of maps.

equivalent to formally inverting the homotopy equivalence.

$$\mathbf{Ho(Top)} \backsim \mathbf{Top}[\text{hoequiv}]^{-1}.$$

Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
  - either a morphism in $\mathcal{C}$
  - or w^{-1}, $w \in \mathcal{W}$.

$Ho(\mathcal{C}) := F(\mathcal{C}, \mathcal{W}^{-1}) / \langle 1 = (1), 1 = (w, w^{-1}), 1 = (w^{-1}, w) \rangle$

Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction: $(L \dashv R) : \mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \underset{\mathbb{L}}{\overset{\mathbb{R}}{\leftrightarrows}} Ho(\mathcal{D})$.

Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
  - either a morphism in $\mathcal{C}$
  - or w^{-1}, $w \in \mathcal{W}$.

$Ho(\mathcal{C}) := F(\mathcal{C}, \mathcal{W}^{-1})/\langle 1 = (1), 1 = (w, w^{-1}), 1 = (w^{-1}, w)\rangle$

Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction: $(L \dashv R) : \mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \underset{\mathbb{L}}{\overset{\mathbb{R}}{\leftrightarrows}} Ho(\mathcal{D})$.

Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
  - either a morphism in $\mathcal{C}$
  - or w^{-1}, $w \in \mathcal{W}$.

$Ho(\mathcal{C}) := F(\mathcal{C}, \mathcal{W}^{-1})/\langle 1 = (1), 1 = (w, w^{-1}), 1 = (w^{-1}, w) \rangle$

Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction: $(L \dashv R) : \mathcal{C} \overset{R}{\underset{L}{\rightleftarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \overset{\mathbb{R}}{\underset{\mathbb{L}}{\rightleftarrows}} Ho(\mathcal{D})$.

Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
  - either a morphism in $\mathcal{C}$
  - or w^{-1}, $w \in \mathcal{W}$.

$Ho(\mathcal{C}) := F(\mathcal{C}, \mathcal{W}^{-1})/\langle 1 = (1), 1 = (w, w^{-1}), 1 = (w^{-1}, w)\rangle$

Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction: $(L \dashv R) : \mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \underset{\mathbb{L}}{\overset{\mathbb{R}}{\leftrightarrows}} Ho(\mathcal{D})$.

Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
  - either a morphism in $\mathcal{C}$
  - or w^{-1}, $w \in \mathcal{W}$.

$Ho(\mathcal{C}) := F(\mathcal{C}, \mathcal{W}^{-1})/\langle 1 = (1), 1 = (w, w^{-1}), 1 = (w^{-1}, w)\rangle$

Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction: $(L \dashv R) : \mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \underset{\mathbb{L}}{\overset{\mathbb{R}}{\leftrightarrows}} Ho(\mathcal{D})$.

Morphisms in the Category of Model Categories

Homotopy Category $Ho(\mathcal{C})$

$\mathcal{C}$: a category. $\mathcal{W}$: a subcategory of weak equivalences.

The free category $F(\mathcal{C}, \mathcal{W}^{-1})$

- same objects as $\mathcal{C}$;
- morphism: a finite string of composable arrows $(f_1, f_2, \cdots f_n)$ where f_i is
  - either a morphism in $\mathcal{C}$
  - or w^{-1}, $w \in \mathcal{W}$.

$$Ho(\mathcal{C}) := F(\mathcal{C}, \mathcal{W}^{-1})/\langle 1 = (1), 1 = (w, w^{-1}), 1 = (w^{-1}, w)\rangle$$

Morphisms $\mathcal{C} \longrightarrow \mathcal{D}$

Quillen adjunction: $(L \dashv R) : \mathcal{C} \underset{L}{\overset{R}{\leftrightarrows}} \mathcal{D}$.

Quillen equivalence: $Ho(\mathcal{C}) \underset{\mathbb{L}}{\overset{\mathbb{R}}{\leftrightarrows}} Ho(\mathcal{D})$.

$G-$CW complex

X^0: disjoint union of orbits G/H.
X^{n+1}: attach $G-$cells $G/H \times D^{n+1}$ to X^n along attaching $G-$maps

$$G/H \times S^n \longrightarrow X^n.$$

Equivariant homotopy group

$$GTop \longrightarrow [Orb_G^{op}, Top]$$
$$X \mapsto (G/H \mapsto X^H)$$

$$\underline{\pi}_n(X)(G/H) = \pi_n(X^H).$$

Elmendorf's theorem

We have the equivalence of the homotopy categories

$$Ho(GTop) \xrightarrow{\simeq} Ho([Orb_G^{op}, Top]).$$

Equivariant homotopy theory

$G-$CW complex

X^0: disjoint union of orbits G/H.
X^{n+1}: attach $G-$cells $G/H \times D^{n+1}$ to X^n along attaching $G-$maps

$$G/H \times S^n \longrightarrow X^n.$$

Equivariant homotopy group

$$GTop \longrightarrow [Orb_G^{op}, Top]$$
$$X \mapsto (G/H \mapsto X^H)$$

$$\pi_n(X)(G/H) = \pi_n(X^H).$$

Elmendorf's theorem

We have the equivalence of the homotopy categories

$$Ho(GTop) \xrightarrow{\simeq} Ho([Orb_G^{op}, Top]).$$

Equivariant homotopy theory

$G-$CW complex

X^0: disjoint union of orbits G/H.
X^{n+1}: attach $G-$cells $G/H \times D^{n+1}$ to X^n along attaching $G-$maps

$$G/H \times S^n \longrightarrow X^n.$$

Equivariant homotopy group

$$GTop \longrightarrow [Orb_G^{op}, Top]$$
$$X \mapsto (G/H \mapsto X^H)$$

$$\underline{\pi}_n(X)(G/H) = \pi_n(X^H).$$

Elmendorf's theorem

We have the equivalence of the homotopy categories

$$Ho(GTop) \xrightarrow{\simeq} Ho([Orb_G^{op}, Top]).$$

Equivariant homotopy theory

$G-$CW complex

X^0: disjoint union of orbits G/H.
X^{n+1}: attach $G-$cells $G/H \times D^{n+1}$ to X^n along attaching $G-$maps

$$G/H \times S^n \longrightarrow X^n.$$

Equivariant homotopy group

$$GTop \longrightarrow [Orb_G^{op}, Top]$$
$$X \mapsto (G/H \mapsto X^H)$$

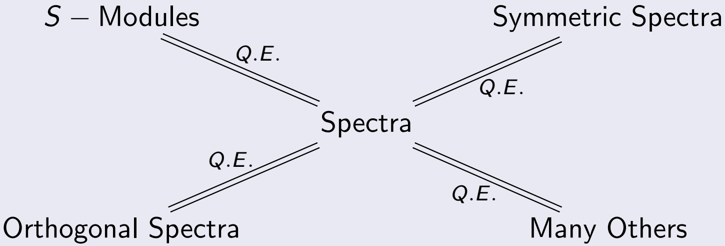
$$\underline{\pi}_n(X)(G/H) = \pi_n(X^H).$$

Elmendorf's theorem

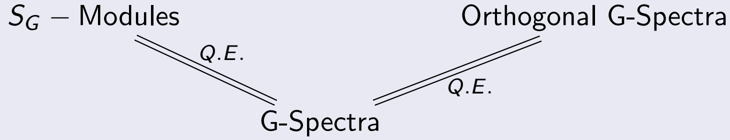
We have the equivalence of the homotopy categories

$$Ho(GTop) \xrightarrow{\simeq} Ho([Orb_G^{op}, Top]).$$

Right Philosophy for Stable Homotopy Theory (1990s) [MMSS]



Equivariant Stable Homotopy Theory (2000s) [MMSS]



Equivariant Stable Homotopy Theory (2000s) [MMSS]

S_G-Modules — Q.E. — G-Spectra — Q.E. — Orthogonal G-Spectra

Which is the BEST model?

Orthogonal $G-$spectra.

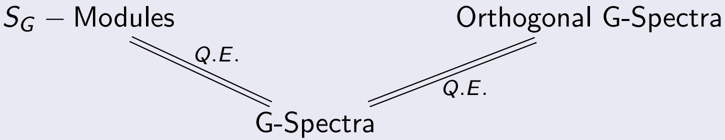
Why BEST?

Combine the best features of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

Model Structures on Equivariant Stable Homotopy Theory

Equivariant Stable Homotopy Theory (2000s) [MMSS]



Which is the BEST model?

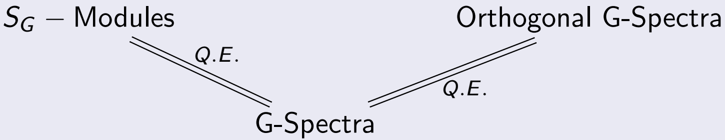
Orthogonal $G-$spectra.

Why BEST?

Combine the best features of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

Equivariant Stable Homotopy Theory (2000s) [MMSS]



Which is the BEST model?

Orthogonal G−spectra.

Why BEST?

Combine the best features of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

Equivariant Stable Homotopy Theory (2000s) [MMSS]

S_G − Modules ═══ *Q.E.* ═══ G-Spectra ═══ *Q.E.* ═══ Orthogonal G-Spectra

Which is the BEST model?

Orthogonal $G-$spectra.

Why BEST?

Combine the best features of other models.

- Coordinate-free.
- Their weak equivalence implies isomorphism of homotopy groups.

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Homotopy groups of an orthogonal $G-$spectrum $\pi_q^H(X)$

$$\pi_q^H(X) = colim_V \; \pi_q^H(\Omega^V X(V)) \;\; \text{if } q \geq 0,$$
$$\pi_{-q}^H(X) = colim_{V \supset \mathbb{R}^q} \; \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \;\; \text{if } q > 0.$$

The weak equivalence of interest: the π_*-isomorphism

$f : X \longrightarrow Y$: induces isomorphisms on all homotopy groups.

$\mathcal{I}_G$: the category of orthogonal representations of G.

Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^- \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Homotopy groups of an orthogonal $G-$spectrum $\pi_q^H(X)$

$$\pi_q^H(X) = colim_V \ \pi_q^H(\Omega^V X(V)) \ \text{ if } q \geq 0,$$

$$\pi_{-q}^H(X) = colim_{V \supset \mathbb{R}^q} \ \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \ \text{ if } q > 0.$$

The weak equivalence of interest: the π_*-isomorphism

$f : X \longrightarrow Y$: induces isomorphisms on all homotopy groups.

$\mathcal{I}_G$: the category of orthogonal representations of G.
Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^- \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Homotopy groups of an orthogonal $G-$spectrum $\pi_q^H(X)$

$$\pi_q^H(X) = colim_V \; \pi_q^H(\Omega^V X(V)) \text{ if } q \geq 0,$$
$$\pi_{-q}^H(X) = colim_{V \supset \mathbb{R}^q} \; \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \text{ if } q > 0.$$

The weak equivalence of interest: the π_*-isomorphism

$f : X \longrightarrow Y$: induces isomorphisms on all homotopy groups.

$\mathcal{I}_G$: the category of orthogonal representations of G.

Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^- \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Homotopy groups of an orthogonal $G-$spectrum $\pi_q^H(X)$

$$\pi_q^H(X) = colim_V \ \pi_q^H(\Omega^V X(V)) \ \text{ if } q \geq 0,$$
$$\pi_{-q}^H(X) = colim_{V \supset \mathbb{R}^q} \ \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \ \text{ if } q > 0.$$

The weak equivalence of interest: the π_*-isomorphism

$f : X \longrightarrow Y$: induces isomorphisms on all homotopy groups.

Orthogonal $G-$spectrum [MMSS]

$\mathcal{I}_G$: the category of orthogonal representations of G.

Top_G: the category of based $G-$spaces and continuous based maps.

$\mathcal{I}_G-$space

A $G-$continuous functor $X : \mathcal{I}_G \longrightarrow Top_G$.

Orthogonal $G-$spectrum

An $\mathcal{I}_G-$space X with a natural transformation $X(-) \wedge S^{-} \longrightarrow X(- \oplus -)$ such that the associativity and unitality diagrams commute.

Homotopy groups of an orthogonal $G-$spectrum $\pi_q^H(X)$

$$\pi_q^H(X) = colim_V \; \pi_q^H(\Omega^V X(V)) \;\; \text{if } q \geq 0,$$
$$\pi_{-q}^H(X) = colim_{V \supset \mathbb{R}^q} \; \pi_0^H(\Omega^{V-\mathbb{R}^q} X(V)) \;\; \text{if } q > 0.$$

The weak equivalence of interest: the π_*-isomorphism

$f : X \longrightarrow Y$: induces isomorphisms on all homotopy groups.

The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG$. $K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG. K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.

$K^0_G(\mathrm{pt}) \cong RG. K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG$. $K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG. K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\text{pt}) \cong RG.K^0_{\mathbb{Z}/n}(\text{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\text{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG. K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

The birth of global homotopy theory

It has been noticed since the beginning of equivariant homotopy theory that certain theories naturally exist not just for a particular group, but in a uniform way for all groups in a specific class.

Example: equivariant K-theory

$K^0_G(X)$: the Grothendieck group of the isomorphism classes of $G-$vector bundles over the $G-$space X.

Example (When G varies)

$K^0_{\{e\}}(X) = K^0(X)$.
$K^0_G(\mathrm{pt}) \cong RG.K^0_{\mathbb{Z}/n}(\mathrm{pt}) \cong \mathbb{Z}[x^{\pm}]/\langle x^n - 1\rangle$. $K^0_{\mathbb{T}}(\mathrm{pt}) \cong \mathbb{Z}[q^{\pm}]$.

Relations between different equivariant K-theories

- Restriction map: $K_G(X) \longrightarrow K_H(X)$;
- Induced map: $K_H(X) \longrightarrow K_G(X)$;
- Change-of-group isomorphism: $K_G(Y \times_H G) \xrightarrow{\cong} K_H(Y)$;

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An **orthogonal space** is a continuous functor from $\mathbb{L}$ to the category of topological spaces.

The category $\mathbb{O}$

- objects: inner product real spaces;
- morphism set $O(V, W)$: the Thom space of the total space

$$\xi(V, W) := \{(w, \phi) \in W \times L(V, W) | w \perp \phi(V)\}$$

of the orthogonal complement vector bundle.

An **orthogonal spectrum** is a based continuous functor from $\mathbb{O}$ to the category of based topological spaces.

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An **orthogonal space** is a continuous functor from $\mathbb{L}$ to the category of topological spaces.

The category $\mathbb{O}$

- objects: inner product real spaces;
- morphism set $O(V, W)$: the Thom space of the total space

$$\xi(V, W) := \{(w, \phi) \in W \times L(V, W) | w \perp \phi(V)\}$$

of the orthogonal complement vector bundle.

An **orthogonal spectrum** is a based continuous functor from $\mathbb{O}$ to the category of based topological spaces.

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An **orthogonal space** is a continuous functor from $\mathbb{L}$ to the category of topological spaces.

The category $\mathbb{O}$

- objects: inner product real spaces;
- morphism set $O(V, W)$: the Thom space of the total space

$$\xi(V, W) := \{(w, \phi) \in W \times L(V, W) | w \perp \phi(V)\}$$

of the orthogonal complement vector bundle.

An **orthogonal spectrum** is a based continuous functor from $\mathbb{O}$ to the category of based topological spaces.

The category $\mathbb{L}$

- objects: inner product real spaces;
- morphism set $L(V, W)$: the linear isometric embeddings.

An **orthogonal space** is a continuous functor from $\mathbb{L}$ to the category of topological spaces.

The category $\mathbb{O}$

- objects: inner product real spaces;
- morphism set $O(V, W)$: the Thom space of the total space

$$\xi(V, W) := \{(w, \phi) \in W \times L(V, W) | w \perp \phi(V)\}$$

of the orthogonal complement vector bundle.

An **orthogonal spectrum** is a based continuous functor from $\mathbb{O}$ to the category of based topological spaces.

Model structures on orthogonal spectra [Schwede]

- level model structure;
- strong level model structure;
- global model structure;

Homotopy groups of an orthogonal spectrum

$$\pi_0^G(X) = \mathrm{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \text{ and } \pi_{-k}^G(X) = \pi_0^G(S^k \wedge X).$$

A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f) : \pi_k^G(X) \xrightarrow{\cong} \pi_k^G(Y)$ for all compact Lie groups G and all integers k.

Model structures on orthogonal spectra [Schwede]

- level model structure;
- strong level model structure;
- global model structure;

Homotopy groups of an orthogonal spectrum

$$\pi_0^G(X) = \text{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \text{ and } \pi_{-k}^G(X) = \pi_0^G(S^k \wedge X).$$

A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f) : \pi_k^G(X) \xrightarrow{\cong} \pi_k^G(Y)$ for all compact Lie groups G and all integers k.

- level model structure;
- strong level model structure;
- global model structure;

Homotopy groups of an orthogonal spectrum

$$\pi_0^G(X) = \mathrm{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \text{ and } \pi_{-k}^G(X) = \pi_0^G(S^k \wedge X).$$

A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f) : \pi_k^G(X) \xrightarrow{\cong} \pi_k^G(Y)$ for all compact Lie groups G and all integers k.

- level model structure;
- strong level model structure;
- global model structure;

Homotopy groups of an orthogonal spectrum

$$\pi_0^G(X) = \mathrm{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \text{ and } \pi_{-k}^G(X) = \pi_0^G(S^k \wedge X).$$

A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f) : \pi_k^G(X) \xrightarrow{\cong} \pi_k^G(Y)$ for all compact Lie groups G and all integers k.

- level model structure;
- strong level model structure;
- global model structure;

Homotopy groups of an orthogonal spectrum

$$\pi_0^G(X) = \text{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \text{ and } \pi_{-k}^G(X) = \pi_0^G(S^k \wedge X).$$

A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f) : \pi_k^G(X) \xrightarrow{\cong} \pi_k^G(Y)$ for all compact Lie groups G and all integers k.

Model structures on orthogonal spectra [Schwede]

- level model structure;
- strong level model structure;
- global model structure;

Homotopy groups of an orthogonal spectrum

$$\pi_0^G(X) = \operatorname{colim}_{V \in s(\mathcal{U}_G)} [S^V, X(V)]^G.$$

$$\pi_k^G(X) = \pi_0^G(\Omega^k X) \text{ and } \pi_{-k}^G(X) = \pi_0^G(S^k \wedge X).$$

A morphism $f : X \longrightarrow Y$ of orthogonal spectra is a **global equivalence** if $\pi_k^G(f) : \pi_k^G(X) \xrightarrow{\cong} \pi_k^G(Y)$ for all compact Lie groups G and all integers k.

Relation with orthogonal $G-$spectra

$$\text{Orthogonal spectra} \underset{\text{arise?}}{\overset{u}{\rightleftarrows}} \text{Equivariant orthogonal spectra}$$

- u: $X \mapsto X\langle G\rangle$ underlying orthogonal $G-$spectrum;

$$\pi_k^G(X) = \pi_k^G(X\langle G\rangle).$$

- arise: $\{X_G\}_G$ with each X_G an orthogonal $G-$spectrum.
  yes iff for any trivial $G-$representation V, the $G-$action on $X_G(V)$ is trivial.

Relation with orthogonal $G-$spectra

$$\text{Orthogonal spectra} \underset{\text{arise?}}{\overset{u}{\rightleftarrows}} \text{Equivariant orthogonal spectra}$$

- u: $X \mapsto X\langle G\rangle$ underlying orthogonal $G-$spectrum;

$$\pi_k^G(X) = \pi_k^G(X\langle G\rangle).$$

- arise: $\{X_G\}_G$ with each X_G an orthogonal $G-$spectrum.
  yes iff for any trivial $G-$representation V, the $G-$action on $X_G(V)$ is trivial.

$$\text{Orthogonal spectra} \underset{\text{arise?}}{\overset{u}{\rightleftarrows}} \text{Equivariant orthogonal spectra}$$

- u: $X \mapsto X\langle G\rangle$ underlying orthogonal $G-$spectrum;
$$\pi_k^G(X) = \pi_k^G(X\langle G\rangle).$$
- arise: $\{X_G\}_G$ with each X_G an orthogonal $G-$spectrum.
yes iff for any trivial $G-$representation V, the $G-$action on $X_G(V)$ is trivial.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Other global homotopy theories

- equivalent to Schwede's global homotopy theory;
- easy to work with for specific theories.

Anna Marie Bohmann: *Global orthogonal spectra*, 2014

- enriched indexed categories;
- Atiyah-Bott-Shapiro orientation has global version.

David Gepner, Andre Henriques: *Homotopy Theory of Orbispaces*, 2007

- infinity categories;
- easier to work with for elliptic cohomology theories.

Zhen Huan: *Almost global homotopy theory*, 2018

- add restriction maps to the category $\mathbb{O}$;
- Quasi-theories can be globalized.

Motivating example: quasi-elliptic cohomology

$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll^*_G(G \times_H X) \cong QEll^*_H(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal $G-$spectrum for $QEll^*_G(-)$, which cannot give a global spectrum in Schwede's sense.
[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Motivating example: quasi-elliptic cohomology

$QEll_G^*(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll_G^*(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll_G^*(G \times_H X) \cong QEll_H^*(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal $G-$spectrum for $QEll_G^*(-)$, which cannot give a global spectrum in Schwede's sense.
[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Motivating example: quasi-elliptic cohomology

$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll^*_G(G \times_H X) \cong QEll^*_H(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal $G-$spectrum for $QEll^*_G(-)$, which cannot give a global spectrum in Schwede's sense.

[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Motivating example: quasi-elliptic cohomology

$QEll^*_G(X) = \prod_{g\in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll^*_G(G \times_H X) \cong QEll^*_H(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal G−spectrum for $QEll^*_G(-)$, which cannot give a global spectrum in Schwede's sense.
[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Motivating example: quasi-elliptic cohomology

$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll^*_G(X) \otimes_{\mathbb{Z}[q^\pm]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll^*_G(G \times_H X) \cong QEll^*_H(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal $G-$spectrum for $QEll^*_G(-)$, which cannot give a global spectrum in Schwede's sense.

[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Motivating example: quasi-elliptic cohomology

$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll^*_G(G \times_H X) \cong QEll^*_H(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal $G-$spectrum for $QEll^*_G(-)$, which cannot give a global spectrum in Schwede's sense.

[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Motivating example: quasi-elliptic cohomology

$QEll^*_G(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll^*_G(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll^*_G(G \times_H X) \cong QEll^*_H(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal $G-$spectrum for $QEll^*_G(-)$, which cannot give a global spectrum in Schwede's sense.

[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Motivating example: quasi-elliptic cohomology

$QEll_G^*(X) = \prod_{g \in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$ [Huan]

- $QEll_G^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) = (K^*_{Tate})_G(X)$;
- Change-of-group isomorphism: $QEll^*_G(G \times_H X) \cong QEll^*_H(X)$.

Question: does global elliptic cohomology theory exist?

- **Jacob Lurie**: Elliptic cohomology theories can be globalized.
- **Nora Ganter**: Quasi-elliptic cohomology has better chances than Grojnowski equivariant elliptic cohomology theory to be put together naturally in a uniform way and made into an ultra-commutative global cohomology theory in the sense of Schwede.
- Cohomology theories with the change-of-group isomorphisms can *PROBABLY* be globalized.

We constructed an orthogonal $G-$spectrum for $QEll^*_G(-)$, which cannot give a global spectrum in Schwede's sense.
[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]

Observation: restriction maps don't need to be identity maps.

- $\{E^n_G, \sigma_{G,n}\}_{n,G}$: equivariant spectra representing $\{E^*_G(-)\}_G$. $E^n_G \simeq_H E^n_H$ for $H \overset{i}{\hookrightarrow} G$.
- For an orthogonal spectrum X, $X(i^*(V)) = i^* X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
- morphisms: $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ with $\phi_2 : V \longrightarrow W$ a linear isometric embedding and $\phi_1 : H \cap O(V) \longrightarrow G$ a group homomorphism.

$$\begin{array}{ccc} G & \longrightarrow & O(V) \\ \uparrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2^*} \\ H \cap O(V) & \longrightarrow & O(W) \end{array}$$

The birth of almost global homotopy theory [Huan]

Observation: restriction maps don't need to be identity maps.

- $\{E^n_G, \sigma_{G,n}\}_{n,G}$: equivariant spectra representing $\{E^*_G(-)\}_G$.
  $E^n_G \simeq_H E^n_H$ for $H \overset{i}{\hookrightarrow} G$.
- For an orthogonal spectrum X, $X(i^*(V)) = i^*X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
- morphisms: $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ with $\phi_2 : V \longrightarrow W$ a linear isometric embedding and $\phi_1 : H \cap O(V) \longrightarrow G$ a group homomorphism.

$$\begin{array}{ccc} G & \longrightarrow & O(V) \\ {\scriptstyle \phi_1}\uparrow & & \downarrow{\scriptstyle \phi_2^*} \\ H \cap O(V) & \longrightarrow & O(W) \end{array}$$

The birth of almost global homotopy theory [Huan]

Observation: restriction maps don't need to be identity maps.

- $\{E^n_G, \sigma_{G,n}\}_{n,G}$: equivariant spectra representing $\{E^*_G(-)\}_G$. $E^n_G \simeq_H E^n_H$ for $H \overset{i}{\hookrightarrow} G$.
- For an orthogonal spectrum X, $X(i^*(V)) = i^*X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
- morphisms: $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ with $\phi_2 : V \longrightarrow W$ a linear isometric embedding and $\phi_1 : H \cap O(V) \longrightarrow G$ a group homomorphism.

$$\begin{array}{ccc} G & \longrightarrow & O(V) \\ \uparrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2^*} \\ H \cap O(V) & \longrightarrow & O(W) \end{array}$$

The birth of almost global homotopy theory [Huan]

Observation: restriction maps don't need to be identity maps.

- $\{E^n_G, \sigma_{G,n}\}_{n,G}$: equivariant spectra representing $\{E^*_G(-)\}_G$.
  $E^n_G \simeq_H E^n_H$ for $H \overset{i}{\hookrightarrow} G$.
- For an orthogonal spectrum X, $X(i^*(V)) = i^*X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
- morphisms: $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ with $\phi_2 : V \longrightarrow W$ a linear isometric embedding and $\phi_1 : H \cap O(V) \longrightarrow G$ a group homomorphism.

$$\begin{array}{ccc} G & \longrightarrow & O(V) \\ \uparrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2^*} \\ H \cap O(V) & \longrightarrow & O(W) \end{array}$$

The birth of almost global homotopy theory [Huan]

Observation: restriction maps don't need to be identity maps.

- $\{E^n_G, \sigma_{G,n}\}_{n,G}$: equivariant spectra representing $\{E^*_G(-)\}_G$.
  $E^n_G \simeq_H E^n_H$ for $H \overset{i}{\hookrightarrow} G$.
- For an orthogonal spectrum X, $X(i^*(V)) = i^*X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
- morphisms: $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ with $\phi_2 : V \longrightarrow W$ a linear isometric embedding and $\phi_1 : H \cap O(V) \longrightarrow G$ a group homomorphism.

$$\begin{array}{ccc} G & \longrightarrow & O(V) \\ \uparrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2^*} \\ H \cap O(V) & \longrightarrow & O(W) \end{array}$$

The birth of almost global homotopy theory [Huan]

Observation: restriction maps don't need to be identity maps.

- $\{E^n_G, \sigma_{G,n}\}_{n,G}$: equivariant spectra representing $\{E^*_G(-)\}_G$.
  $E^n_G \simeq_H E^n_H$ for $H \overset{i}{\hookrightarrow} G$.
- For an orthogonal spectrum X, $X(i^*(V)) = i^*X(V)$ for any $G-$representation V.

The new diagram D_0: add restriction maps to $\mathbb{L}$

- objects: (G, V) with $G \leqslant O(V)$ finite
- morphisms: $\phi = (\phi_1, \phi_2) : (G, V) \longrightarrow (H, W)$ with $\phi_2 : V \longrightarrow W$ a linear isometric embedding and $\phi_1 : H \cap O(V) \longrightarrow G$ a group homomorphism.

$$\begin{array}{ccc} G & \longrightarrow & O(V) \\ \uparrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2^*} \\ H \cap O(V) & \longrightarrow & O(W) \end{array}$$

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
    - linear isometric embedding: raising degree;
    - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0−space and D_0−spectrum.

A D_0−space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0−spectrum X consists of

- a based G−space $X(G, V)$;
- a $G \times H$−equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0−spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0−space and D_0−spectrum.

A D_0−space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0−spectrum X consists of

- a based G−space $X(G, V)$;
- a $G \times H$−equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0−spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.
A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
    - linear isometric embedding: raising degree;
    - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The feature of D_0

- D_0 is a symmetric monoidal category.
- D_0 is a generalized Reedy category in Berger and Moerdijk's sense.
  - linear isometric embedding: raising degree;
  - restriction map: lowering degree.

We can also define D_0-space and D_0-spectrum.

A D_0-space is a continuous functor from D_0 to the category of based compactly generated weak Hausdorff spaces.

A D_0-spectrum X consists of

- a based $G-$space $X(G, V)$;
- a $G \times H-$equivariant based structure map $\sigma_{(G,V),(H,W)} : S^W \wedge X(G, V) \longrightarrow X(G \times H, V \oplus W)$

A morphism of D_0-spectra: compatible with the structure maps.

But these are NOT the right subjects to study.

The category $D_0 T^W$

the full subcategory of $D_0 T$ consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an H−weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W−spectra

A D_0^W−spectrum X is both a D_0−spectrum and a D_0−space in $D_0 T^W$.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

• The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−level model structure on orthogonal spectra.
• The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−global model structure on orthogonal spectra.

[Zhen Huan: *Almost global homotopy theory*, 2018]

The right subject: D_0^W−spectra [Huan]

The category $D_0 T^W$

the full subcategory of $D_0 T$ consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an H−weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W−spectra

A D_0^W−spectrum X is both a D_0−spectrum and a D_0−space in $D_0 T^W$.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

- The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−level model structure on orthogonal spectra.
- The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−global model structure on orthogonal spectra.

[Zhen Huan: *Almost global homotopy theory*, 2018]

The right subject: D_0^W−spectra [Huan]

The category $D_0 T^W$

the full subcategory of $D_0 T$ consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an H−weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W−spectra

A D_0^W−spectrum X is both a D_0−spectrum and a D_0−space in $D_0 T^W$.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

- The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−level model structure on orthogonal spectra.
- The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−global model structure on orthogonal spectra.

[Zhen Huan: *Almost global homotopy theory*, 2018]

The right subject: D_0^W-spectra [Huan]

The category D_0T^W

the full subcategory of D_0T consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an $H-$weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W-spectra

A D_0^W-spectrum X is both a D_0-spectrum and a D_0-space in D_0T^W.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

- The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the $Fin-$level model structure on orthogonal spectra.
- The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the $Fin-$global model structure on orthogonal spectra.

[Zhen Huan: *Almost global homotopy theory*, 2018]

The right subject: D_0^W−spectra [Huan]

The category $D_0 T^W$

the full subcategory of $D_0 T$ consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an H−weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W−spectra

A D_0^W−spectrum X is both a D_0−spectrum and a D_0−space in $D_0 T^W$.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

- The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−level model structure on orthogonal spectra.
- The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−global model structure on orthogonal spectra.

[Zhen Huan: *Almost global homotopy theory*, 2018]

The category $D_0 T^W$

the full subcategory of $D_0 T$ consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an H−weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W−spectra

A D_0^W−spectrum X is both a D_0−spectrum and a D_0−space in $D_0 T^W$.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

• The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−level model structure on orthogonal spectra.
• The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the *Fin*−global model structure on orthogonal spectra.

[Zhen Huan: *Almost global homotopy theory*, 2018]

The category $D_0 T^W$

the full subcategory of $D_0 T$ consisting of those objects $X : D_0 \longrightarrow T$ that maps each restriction map $(G, V) \longrightarrow (H, V)$ to an $H-$weak equivalence.

$Sp_W^{D_0}$: the category of D_0^W-spectra

A D_0^W-spectrum X is both a D_0-spectrum and a D_0-space in $D_0 T^W$.

Relation with Schwede's global homotopy theory

$$(P \dashv Q) : Sp^O \underset{P}{\overset{Q}{\rightleftarrows}} Sp_W^{D_0}$$

• The Reedy model structure on $Sp_W^{D_0}$ is Quillen equivalent to the $Fin-$level model structure on orthogonal spectra.
• The global model structure on $Sp_W^{D_0}$ is Quillen equivalent to the $Fin-$global model structure on orthogonal spectra.

[Zhen Huan: *Almost global homotopy theory*, 2018]

Quasi-theories [Huan]

Divisible group: a better algebraic object associated to an elliptic curve than formal group.

the generalized Tate K-theory and generalized quasi-elliptic cohomology

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_n \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}^n} \mathbb{Q}^n)[p^k]) = K^0_{n,Tate}(B\mathbb{Z}_{p^k})$.
The corresponding quasi-theory:
$QK^*_{n,G}(X) := K^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma \in G^n_z} K^*_{\Lambda^n_G(\sigma)}(X^\sigma)$.

$$K^*_{n,Tate\,G}(X) \cong QK^*_{n,G}(X) \otimes_{\mathbb{Z}[q^\pm]^{\otimes n}} \mathbb{Z}((q))^{\otimes n}.$$

Quasi-theories

$QE^*_{n,G}(X) := E^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma \in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma)$.

[Zhen Huan: *Quasi-theories*, 2018]

Divisible group: a better algebraic object associated to an elliptic curve than formal group.

the generalized Tate K-theory and generalized quasi-elliptic cohomology

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_n \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}^n} \mathbb{Q}^n)[p^k]) = K^0_{n,Tate}(B\mathbb{Z}_{p^k})$.
The corresponding quasi-theory:
$QK^*_{n,G}(X) := K^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma \in G^n_z} K^*_{\Lambda^n_G(\sigma)}(X^\sigma)$.

$$K^*_{n,Tate\,G}(X) \cong QK^*_{n,G}(X) \otimes_{\mathbb{Z}[q^\pm]^{\otimes n}} \mathbb{Z}((q))^{\otimes n}.$$

Quasi-theories

$QE^*_{n,G}(X) := E^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma \in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma)$.

[Zhen Huan: *Quasi-theories*, 2018]

Divisible group: a better algebraic object associated to an elliptic curve than formal group.

the generalized Tate K-theory and generalized quasi-elliptic cohomology

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_n \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}^n} \mathbb{Q}^n)[p^k]) = K^0_{n,Tate}(B\mathbb{Z}_{p^k})$.

The corresponding quasi-theory:

$$QK^*_{n,G}(X) := K^*(\Lambda^n(X /\!\!/ G)) \cong \prod_{\sigma \in G^n_z} K^*_{\Lambda^n_G(\sigma)}(X^\sigma).$$

$$K^*_{n,Tate\,G}(X) \cong QK^*_{n,G}(X) \otimes_{\mathbb{Z}[q^{\pm}]^{\otimes n}} \mathbb{Z}((q))^{\otimes n}.$$

Quasi-theories

$$QE^*_{n,G}(X) := E^*(\Lambda^n(X /\!\!/ G)) \cong \prod_{\sigma \in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma).$$

[Zhen Huan: *Quasi-theories*, 2018]

Divisible group: a better algebraic object associated to an elliptic curve than formal group.

the generalized Tate K-theory and generalized quasi-elliptic cohomology

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_n \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}^n} \mathbb{Q}^n)[p^k]) = K^0_{n, Tate}(B\mathbb{Z}_{p^k})$.
The corresponding quasi-theory:
$QK^*_{n,G}(X) := K^*(\Lambda^n(X/\!/G)) \cong \prod_{\sigma \in G^n_z} K^*_{\Lambda^n_G(\sigma)}(X^\sigma)$.

$$K^*_{n, Tate\, G}(X) \cong QK^*_{n,G}(X) \otimes_{\mathbb{Z}[q^\pm]^{\otimes n}} \mathbb{Z}((q))^{\otimes n}.$$

Quasi-theories

$$QE^*_{n,G}(X) := E^*(\Lambda^n(X/\!/G)) \cong \prod_{\sigma \in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma).$$

[Zhen Huan: *Quasi-theories*, 2018]

Quasi-theories [Huan]

Divisible group: a better algebraic object associated to an elliptic curve than formal group.

the generalized Tate K-theory and generalized quasi-elliptic cohomology

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_n \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}^n} \mathbb{Q}^n)[p^k]) = K^0_{n,Tate}(B\mathbb{Z}_{p^k})$.
The corresponding quasi-theory:
$QK^*_{n,G}(X) := K^*(\Lambda^n(X /\!\!/ G)) \cong \prod_{\sigma \in G^n_z} K^*_{\Lambda^n_G(\sigma)}(X^\sigma)$.

$$K^*_{n,Tate_G}(X) \cong QK^*_{n,G}(X) \otimes_{\mathbb{Z}[q^\pm]^{\otimes n}} \mathbb{Z}((q))^{\otimes n}.$$

Quasi-theories

$$QE^*_{n,G}(X) := E^*(\Lambda^n(X /\!\!/ G)) \cong \prod_{\sigma \in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma).$$

[Zhen Huan: *Quasi-theories*, 2018]

Divisible group: a better algebraic object associated to an elliptic curve than formal group.

the generalized Tate K-theory and generalized quasi-elliptic cohomology

$$0 \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_n \longrightarrow (\mathbb{Q}/\mathbb{Z})^n \longrightarrow 0.$$

$\mathbb{G}_m$: formal group of Tate K-theory; $\Gamma((\mathbb{G}_m \oplus_{\mathbb{Z}^n} \mathbb{Q}^n)[p^k]) = K^0_{n,Tate}(B\mathbb{Z}_{p^k})$.
The corresponding quasi-theory:
$QK^*_{n,G}(X) := K^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma\in G^n_z} K^*_{\Lambda^n_G(\sigma)}(X^\sigma)$.

$$K^*_{n,Tate\,G}(X) \cong QK^*_{n,G}(X) \otimes_{\mathbb{Z}[q^\pm]^{\otimes n}} \mathbb{Z}((q))^{\otimes n}.$$

Quasi-theories

$QE^*_{n,G}(X) := E^*(\Lambda^n(X/\!\!/G)) \cong \prod_{\sigma\in G^n_z} E^*_{n,\Lambda^n_G(\sigma)}(X^\sigma)$.

[Zhen Huan: *Quasi-theories*, 2018]

Theorem

If the theory $\{E^*_G(-)\}_G$ can be globalized, there is a D_0^W-spectrum representing the quasi-theory $\{QE^*_{n,G}(-)\}_G$.
In particular, quasi-elliptic cohomology, the quasi-theory of Tate K-theory, can be globalized in almost global homotopy theory.

[Zhen Huan: *Quasi-elliptic cohomology*, PhD thesis]
[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]
[Zhen Huan: *Quasi-theories and their equivariant orthogonal spectra*, 2018]
[Zhen Huan: *Almost global homotopy theory*, 2018]

My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

Theorem

If the theory $\{E^*_G(-)\}_G$ can be globalized, there is a D^W_0-spectrum representing the quasi-theory $\{QE^*_{n,G}(-)\}_G$.
In particular, quasi-elliptic cohomology, the quasi-theory of Tate K-theory, can be globalized in almost global homotopy theory.

[Zhen Huan: *Quasi-elliptic cohomology*, PhD thesis]
[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]
[Zhen Huan: *Quasi-theories and their equivariant orthogonal spectra*, 2018]
[Zhen Huan: *Almost global homotopy theory*, 2018]

My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

Theorem

If the theory $\{E^*_G(-)\}_G$ can be globalized, there is a D_0^W-spectrum representing the quasi-theory $\{QE^*_{n,G}(-)\}_G$.
In particular, quasi-elliptic cohomology, the quasi-theory of Tate K-theory, can be globalized in almost global homotopy theory.

[Zhen Huan: *Quasi-elliptic cohomology*, PhD thesis]
[Zhen Huan: *Quasi-elliptic cohomology and its Spectrum*, 2017]
[Zhen Huan: *Quasi-theories and their equivariant orthogonal spectra*, 2018]
[Zhen Huan: *Almost global homotopy theory*, 2018]

My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

Thank you.

Some references

https://huanzhen84.github.io/zhenhuan/Huan-Fudan-2018.pdf

- Ando, Hopkins, Strickland: "Elliptic spectra, the Witten genus and the theorem of the cube". Invent. Math. , 146(3):595–687, 2001.
- Ando, Hopkins, Strickland, "The sigma orientation is an H_∞ map", Amer. J. 2004;
- Atiyah, "Power operations in K-theory", Quart. J. Math. Oxford Ser. (2) 17 1966.
- Atiyah,"Equivariant K-theory and completion", J. Differential Geometry 3 1969.
- Berger, Moerdijk, "On an extension of the notion of Reedy category", Mathematische Zeitschrift, December 2011.
- Bohmann, "Global orthogonal spectra", Homology, Homotopy and Applications. Vol 16 (2014), No 1, 313–332.
- Gepner, "Homotopy Topoi and Equivariant Elliptic Cohomology", Thesis (Ph.D.)University of Illinois at Urbana-Champaign. 1999.
- Gepner, Henriques, "Homotopy Theory of Orbispaces", available at arXiv:math/0701916.
- Ginzburg, Kapranov, Vasserot, "Elliptic algebras and equivariant elliptic cohomology I", available at arXiv:q-alg/9505012.
- Greenlees, May, "Localization and completion theorems for MU- module spectra", Ann. of Math. (2), 146(3) (1997), 509–544.
- Katz, Mazur, "Arithmetic moduli of elliptic curves", Annals of Mathematics Studies, vol. 108, 1985.
- Lurie, "A Survey of Elliptic Cohomology", in Algebraic Topology Abel Symposia Volume 4, 2009, pp 219–277.
- Mandell, May, "Equivariant orthogonal spectra and S-modules", Mem.,Amer. Math. Soc. 159 (2002), no. 755, x+108 pp.
- Mandell, May, Schwede, Shipley, "Model categories of diagram spectra", Proc. London Math. Soc. 82(2001).
- May, "Equivariant homotopy and cohomology theory", CBMS Regional Conference Series in Mathematics, vol. 91, 1996.
- Rezk, "Quasi-elliptic cohomology", 2011.
- Schwede, "Global Homotopy Theory", global.pdf.