

Quasi-elliptic cohomology

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December 23, 2018

Plan.

- Motivation and Definition
- Loop space construction: Witten's conjecture
- Geometric structure on Tate curve
- Global homotopy theory for elliptic cohomology theories
- Further problems
 - equivariant homotopy theory
 - chromatic homotopy theory
 - motivic homotopy theory

generalized cohomology theories $\xrightarrow{\text{1st Chern class}}$ formal groups
 \longleftarrow

even periodic, multiplicative

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

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commutative 1-dimensional formal groups

- The additive formal group \mathbb{G}_a : periodic Eilenberg-MacLane spectrum.

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- The multiplicative formal group \mathbb{G}_m : complex K-theory.

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- Elliptic curves: elliptic cohomology?

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- The multiplicative formal group \mathbb{G}_m : complex K-theory.
- Elliptic curves: elliptic cohomology?

Elliptic cohomology

[AHS][Lurie]

R : commutative ring; C/R : elliptic curve over R .

E is an elliptic cohomology theory if $E^0(\text{pt}) \cong R$ and $\text{Spf}E^0(\mathbb{C}P^\infty) \cong \widehat{C}$.

Moduli stack of elliptic curves

$$\{\text{elliptic curves } E \longrightarrow \text{Spec}R\} = \text{Hom}(\text{Spec}R, M_{1,1}).$$

{*et*alé elliptic curves over R } \longrightarrow {multiplicative cohomology theories}.

Tate K-theory

[AHS]

Tate curve: the generalized elliptic curve over $\mathbb{Z}((q))$ classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.

The associated formal group: \mathbb{G}_m over $\mathbb{Z}((q))$.

Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Good Features:

- relation with K-theory.
- relation with string theory;
- relation with loop space.

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Explicit Definition

$$QEII_G^*(X) \cong \prod_{g \in G_{conj}^{tors}} K_{\Lambda_G(g)}^*(X^g)$$

- G_{conj}^{tors} : a set of representatives of G -conjugacy classes in G^{tors} ;
- $\Lambda_G(g) = C_G(g) \times \mathbb{R} / \langle (g, -1) \rangle$;
- $x \cdot [a, t] = x \cdot a$, for all $[a, t] \in \Lambda_G(g)$, $x \in X^g$.

Relation with Tate K-theory

$$QEII_G^*(X) \otimes_{\mathbb{Z}[q^{\pm 1}]} \mathbb{Z}((q)) \cong K_{Tate}^*(X // G).$$

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Representation theory

- Restriction map: $RG \longrightarrow RH$;

Equivariant K-theory

- Restriction map: $K_G^*(X) \longrightarrow K_H^*(X)$;

Quasi-elliptic cohomology

- Restriction map: $QEll_G^*(X) \longrightarrow QEll_H^*(X)$;

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- $RG \otimes RH \longrightarrow R(G \times H)$.

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- $K_G^*(-)$ can be represented by an orthogonal G -spectrum;

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- Change-of-group isomorphism: $K_G^*(Y \times_H G) \xrightarrow{\cong} K_H^*(Y)$;
- $K_G^*(-)$ can be represented by an orthogonal G -spectrum;
- **Global K-theory exists.**

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An old idea by Witten

[Landweber]

$$LX = \mathbb{C}^\infty(\mathbb{T}, X),$$

$$EII^*(X) \overset{?}{\longleftrightarrow} K_{\mathbb{T}}^*(LX)$$

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Relevant Work

[Devoto][Ganter]

2007, G -equivariant Tate K-theory for finite groups G is modelled on the loop space of a global quotient orbifold.

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Question

How can we construct elliptic cohomology theories from loop spaces?

Review: Free Loop Space

$$LX = C^\infty(\mathbb{T}, X).$$

- \mathbb{T} -action: $\gamma \cdot t = (s \mapsto \gamma(s + t))$.
- LG -action: $\gamma \cdot \delta = (s \mapsto \gamma(s) \cdot \delta(s))$.
- $LG \rtimes \mathbb{T}$ -action: $\gamma \cdot (\delta, t) = (s \mapsto \gamma(s + t) \cdot \delta(s + t))$.
 $(\delta_1, t_1) \cdot (\delta_2, t_2) = (s \mapsto \delta_1(s)\delta_2(s + t_1), t_1 + t_2)$.

Interpretation of the $LG \rtimes \mathbb{T}$ -action

LG : the gauge group of the trivial G -bundle over \mathbb{T} .

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$LG \rtimes \mathbb{T}$: the extended gauge group

$$\begin{array}{ccc}
 G \times \mathbb{T} & \xrightarrow{(g,s) \mapsto (\gamma(s)g, s+t)} & G \times \mathbb{T} \\
 \downarrow & & \downarrow \\
 \mathbb{T} & \xrightarrow{s \mapsto s+t} & \mathbb{T}
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$LG \rtimes \mathbb{T}$: act on loops $G \times \mathbb{T} \longrightarrow G \times \mathbb{T} \xrightarrow{\tilde{\gamma}} X$

$$\begin{array}{ccc} G \times \mathbb{T} & \longrightarrow & G \times \mathbb{T} \\ \downarrow & & \downarrow \\ \mathbb{T} & \longrightarrow & \mathbb{T} \end{array}$$

The Answer: What is "Loop"?

New Definition of Equivariant loops $Loop(X//G)$

[Rezk]

- Objects:

$$\mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X$$

- π : principal G -bundle over \mathbb{T}
- f : G -equivariant;
- Morphism $(\alpha, t) : \{ \mathbb{T} \xleftarrow{\pi} P' \xrightarrow{f'} X \} \rightarrow \{ \mathbb{T} \xleftarrow{\pi} P \xrightarrow{f} X \}$:

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Relation with Bibundles

$Bibun(\mathbb{T} // *, X // G)$

- same objects;
- morphisms: (α, Id) . No rotations.

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$\Lambda(X//G)$: a subgroupoid of $Loop(X//G)$ consisting of constant loops.

$$\Lambda(X//G) \cong \coprod_{g \in G_{conj}^{tors}} X^g // \Lambda_G(g)$$

$$QEII_G^*(X) = K_{orb}^*(\Lambda(X//G))$$

Further Question

Can we use this new definition of loop spaces to construct elliptic cohomology theories?

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Classification Theorems on elliptic curves

Motivating examples: Morava E-theories

- 1995, Matthew Ando: a classification of the level- p^k structure of its formal group.
- 1998, Neil Strickland: $\text{Spec}(E^0(B\Sigma_{p^k})/I_{tr}) \cong \text{Sub}_{p^k}(\mathbb{G}_E)$.
The Morava E -theory of the symmetric group Σ_n modulo a certain transfer ideal classifies the power subgroups of rank n of its formal group.
- 2015, Tomer M. Schlank, Nathaniel Stapleton:

$$\text{Spec}(E^0(L^h(B\Sigma_{p^k}))/I_{tr}) \cong \text{Sub}_{p^k}(\mathbb{G}_E \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h).$$

Via transchromatic character theory.

the homotopy theory $\xleftrightarrow{\text{Power Operation}}$ its formal group.

Question

Are the geometric structures on elliptic curves classified in the same way?

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Power Operation of K-theory

[Atiyah]

$$P_n : K(X) \longrightarrow K_{\Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Power Operation of equivariant K-theory

[Atiyah]

$$P_n : K_G(X) \longrightarrow K_{G \wr \Sigma_n}(X^{\times n}), \quad V \mapsto V^{\boxtimes n}$$

Wreath product $G \wr \Sigma_n$

$$(g_1, \dots, g_n, \sigma) \cdot (h_1, \dots, h_n, \tau) := (g_1 h_{\sigma^{-1}(1)}, \dots, g_n h_{\sigma^{-1}(n)}, \sigma\tau).$$

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Atiyah's Power Operation

[Ganter]

V : a vector bundle over $\Lambda(X//G)$.

$P_n(V) := V^{\widehat{\otimes}_{\mathbb{Z}[q^{\pm}]}^n}$ defines an operation

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The Stringy Power Operation

[Huan]

$$\mathbb{P}_n = \prod_{(\underline{g}, \sigma) \in (G|\Sigma_n)_{conj}^{tors}} \mathbb{P}_{(\underline{g}, \sigma)}:$$

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Theorem (Huan)

The Tate K -theory of symmetric groups modulo a certain transfer ideal classifies finite subgroups of the Tate curve.

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where $I_{\text{tr}}^{\text{Tate}}$ is the transfer ideal and q'_s is the image of q under the **stringy power operation**, the product goes over all the ordered pairs of positive integers (d, e) such that $N = de$.

The proof: (i) Apply representation theory, prove

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Quasi-theories

[Huan]

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My conjecture

The globalness of a cohomology theory is determined by the formal component of its divisible group; when the étale component varies, the globalness does not change.

More explicitly, if E^* can be globalized and A^* is a cohomology theory with divisible group $\mathbb{G}_E \oplus (\mathbb{Q}/\mathbb{Z})^n$, then A^* can also be globalized.

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Future Problems in chromatic homotopy theory

Chromatic level:=height of the formal group.

Chromatic level	complex oriented cohomology theory
0	ordinary cohomology 0th Morava K-theory $K(0)$
1	complex K-theory KU first Morava K-theory $K(1)$ first Morava E-theory $E(1)$
2	elliptic cohomology second Morava K-theory $K(2)$ second Morava E-theory $E(2)$
n	n th Morava K-theory $K(n)$ n th Morava E-theory $E(n)$
∞	complex bordism cohomology MU .

Table: Chromatic homotopy theory

Interesting point

The chromatic level of elliptic cohomology theories is **2**;
the chromatic level of quasi-elliptic cohomology theory is **1**.

Question: How can we relate the two closed theories?

Transchromatic character theory

- The character map: $L \otimes R(G) \xrightarrow{\cong} CI(G; L)$.
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Gronowski's elliptic cohomology theories and quasi-elliptic cohomology theory can be related by Stapleton's generalized character map.

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Generalized Morava E -theories have Strickland's theorem for A -level structures. Explicitly,

$$\mathrm{Spec}(E_n^0(L^h(BA))/I_{tr}^A) = \mathrm{Level}_A(\mathbb{G}_{E_n} \oplus (\mathbb{Q}_p/\mathbb{Z}_p)^h)$$

- A : any finite abelian group;
- $L(-)$: the p -adic loop functor $\mathrm{Hom}(*//\mathbb{Z}_p, X)$.

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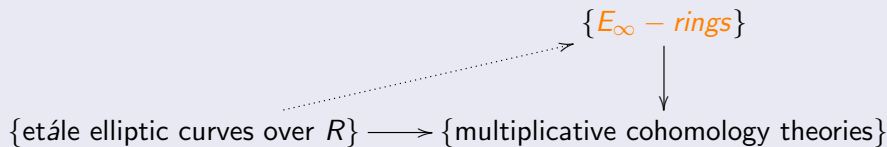
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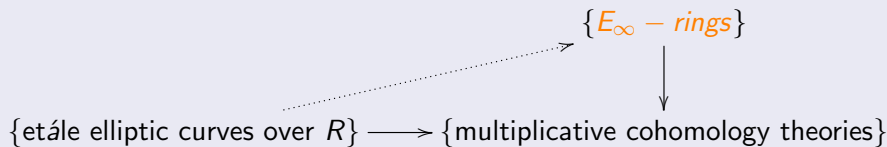
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Especially, motivic elliptic cohomology theories exist.
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<http://gagp.sysu.edu.cn/zhenhuan/Zhen-HUST-2018-Slides.pdf>

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