Quasi-Elliptic Cohomology

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Introduction and Definition

Quasi-elliptic cohomology is motivated by Nora Ganter; set up by Charles Rezk; developed by Zhen Huan.

Tate K-theory

- Tate curve: classified as the completion of the algebraic stack of some nice generalized elliptic curves at infinity.
- Tate K-theory: generalized elliptic cohomology associated to the Tate curve.

Quasi-elliptic cohomology

- A variant of Tate K-theroy.
- Not an elliptic cohomology.
- Geometric features of Tate curve.
- Neat form.

Expression

$$QEll^*_G(X) = \prod_{g\in G^{tors}_{conj}} K^*_{\Lambda_G(g)}(X^g)$$

- X: compact $G-$space;
- G^{tors}_{conj}: a set of representatives of $G-$conjugacy classes in G^{tors};
- $\Lambda_G(g) = C_G(g)\times\mathbb{R}/\langle(g,-1)\rangle$;
- $\Lambda_G(g)$ acts on X^g by $[h,t]\cdot x := h\cdot x$.

Relation with Tate K-theory

$$QEll^*_G(X)\otimes_{\mathbb{Z}[q^{\pm}]}\mathbb{Z}((q)) \cong K^*_{Tate}(X/\!/G).$$

Properties

- Restriction map: $QEll_G(X)\longrightarrow QEll_H(X)$;
- Künneth map: $QEll^*_G(X)\widehat{\otimes}_{\mathbb{Z}[q]}QEll^*_H(Y)\longrightarrow QEll^*_{G\times H}(X\times Y)$;
- Change-of-group isomorphism: $QEll_G(Y\times_H G)\xrightarrow{\cong} QEll_H(Y)$;
- Induced map: $QEll_H(X)\longrightarrow QEll_G(X)$;

Loop Space Construction [Huan]

An old idea by Witten

$LX = \mathbb{C}^\infty(S^1, X)$, $\mathbb{T}$ acts on S^1, G acts on X.

$$Ell^*(X) \overset{?}{\rightsquigarrow} K^*_{\mathbb{T}}(LX)$$

Construction

Loops$\sim$Bibundles: 1-morphisms in $Gpd[W^{-1}]$.

$Loop(X/\!/G) := Bibun(S^1/\!/*, X/\!/G)$

- Objects: $\mathcal{P} := \{S^1 \xleftarrow{\pi} P \xrightarrow{f} X\}$
  - π : principal $G-$bundle over S^1
  - f : $G-$equivariant;
- Morphism $\mathcal{P}\longrightarrow\mathcal{P}'$:
  - $G-$bundle map $\alpha : P\longrightarrow P'$

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ & \nwarrow^{\pi'} & \downarrow{\scriptstyle\alpha} & \nearrow_{f'} & \\ & & P' & & \end{array}$$

$Loop^{ext}(X/\!/G)$: Add the rotation

- Objects: SAME.
- Morphism $\mathcal{P}\longrightarrow\mathcal{P}'$: (α, t).
  - $G-$bundle map $\alpha : P\longrightarrow P'$,
  - $t\in\mathbb{R}$.

$$\begin{array}{ccccc} S^1 & \xleftarrow{\pi} & P & \xrightarrow{f} & X \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle\alpha} & \nearrow_{f'} & \\ S^1 & \xleftarrow{\pi'} & P' & & \end{array}$$

$\Lambda(X/\!/G)$: a groupoid of constant loops

A subgroupoid of $Loop^{ext}(X/\!/G)$

$$\Lambda(X/\!/G) := \coprod_{g\in G^{tors}_{conj}} X^g/\!/\Lambda_G(g)$$

$$QEll_G(X)\cong K_{orb}(\Lambda(X/\!/G))$$

How Quasi-elliptic cohomology reflects geometric nature of Tate curve

History

- 1995, Matthew Ando, Neil Strickland: $E^0(\bigvee_{k<0}B\Sigma_{k+})\rightsquigarrow$ the subgroups of the formal group.
- 1998, Neil Strickland: $\mathrm{Spec}(E^0(B\Sigma_{p^k})/I_{tr})\cong Sub_{p^k}(\mathbb{G}_E)$.
- 2015, Tomer M.Schlank, Nathaniel Stapleton: $\mathrm{Spec}(E^0(L^hB\Sigma_{p^k})/I_{tr})\cong Sub_{p^k}(\mathbb{G}_E\oplus(\mathbb{Q}_p/\mathbb{Z}_p)^h)$.

the homotopy theory $\xrightarrow{\text{Power Operation}}$ its formal group.

Power operation of Quasi-elliptic cohomology theory [Huan]

- Atiyah's Power operation, as that of equivariant K-theories.
- an elliptic power operation $\rightsquigarrow$ stringy power operation of Tate K-theory.

Classification of the finite subgroups of Tate curve [Huan]

$\mathrm{Spec}(K^0_{Tate}(\mathrm{pt}/\!/\Sigma_N)/I_{tr})\cong Sub_N(Tate(q))$ for any integer N.

Classification of $A-$level structures of Tate curve [Huan]

$$\mathrm{Spec}(K^0_{Tate}(\mathrm{pt}/\!/A)/I^A_{tr})\cong Level_A(Tate(q))$$

for any finite abelian group A.

The role of quasi-elliptic cohomology theory

Reduce the problems into questions in representation theory.

The idea of the proof

- $QEll^0_{\Sigma_N}(\mathrm{pt}) = \prod_{g\in\Sigma^{tors}_{N\,conj}} K_{\Lambda_G(g)}(\mathrm{pt}) = \prod_{g\in\Sigma^{tors}_{N\,conj}} R\Lambda_G(g).$
- Use representation theory to compute

$$QEll^0_{\Sigma_N}(\mathrm{pt})/I_{tr}\cong\prod_{N=de}\mathbb{Z}[q^{\pm}][q']/\langle q^d - q'^e\rangle.$$

- Apply the relation $QEll^*_G(X)\otimes_{\mathbb{Z}[q^{\pm}]}\mathbb{Z}((q))\cong K^*_{Tate}(X/\!/G)$.

The Spectra of Equivariant Elliptic Cohomologies

Ginzburg, Kapranov and Vasserot's Conjecture

Any elliptic curve A gives rise to a unique equivariant elliptic cohomology theory, natural in A.

Orthogonal $G-$spectrum of quasi-elliptic cohomology [Huan]

We construct explicitly a commutative $\mathcal{I}_G-$FSP $(E(G,-),\eta,\mu)$ with G a compact Lie group. It weakly represents $QEll^V_G(-)$ in the sense

$$\pi_0(E(G,V)) = QEll^V_G(S^0)$$

for each faithful $G-$representation V.

The construction can be applied to generalized Morava E-theory and equivariant Tate K-theory.

Can $E(G,-)$ arise from an orthogonal spectrum?

No.

For a trivial $G-$representation V, the $G-$action on $E(G,V)$ is not trivial.

Then, it's even more difficult for equivariant elliptic cohomology to fit into the global homotopy theory!

A new global homotopy theory [Huan]

Features

- Contains all the global homotopy theories;
- Global quasi-elliptic cohomology can be defined;
- The new and old global homotopy theories describe the same mathematical world.

The idea: we still use diagram spectra.

The category D_0: add restriction maps to the linear isometries category $\mathbb{L}$

- Objects: (G, V)
  - G: a finite group;
  - V: a faithful $G-$representation;
- Morphisms: $\phi = (\phi_1,\phi_2) : (G,V)\longrightarrow(H,W)$
  - a linear isometric embedding $\phi_2 : V\longrightarrow W$;
  - a group homomorphism $\phi_1 : H\cap\phi_{2*}(\Sigma_{|dimV|})\longrightarrow G$;

$$\begin{array}{ccc} G & \longrightarrow & \Sigma_{|dimV|} \\ \uparrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_{2*}} \\ H\cap\phi_{2*}(\Sigma_{|dimV|}) & \longrightarrow & \Sigma_{|dimW|} \end{array}$$

This is a generalized Reedy category.

- linear isometric embedding: raising degree;
- restriction map: lowering degree.

The category of global spaces: D_0T^W

- D_0T: the category of D_0-spaces;
- D_0T^W: the full subcategory of D_0T consisting of $X : D_0\longrightarrow T$ that maps each restriction map $(G,V)\longrightarrow(H,V)$ to an $H-$weak equivalence.

The relation between D_0T^W and $\mathbb{L}T$

Define

$$R : D_0T^W\xrightarrow{incl} D_0T\xrightarrow{f^*}\mathbb{L}T \text{ with } f:\mathbb{L}\longrightarrow D_0,\ V\mapsto(\Sigma_{|dimV|}, V)$$

We have a pair of adjoint functors

$$(L\dashv R) : \mathbb{L}T \underset{L}{\overset{R}{\rightleftarrows}} D_0T^W$$

- The Reedy model structure on D_0T^W is Quillen equivalent to the $\mathcal{F}in-$level model structure on $\mathbb{L}T$.
- The global model structure on D_0T^W is Quillen equivalent to the $\mathcal{F}in-$global model structure on $\mathbb{L}T$.

Future Problems

- Construct the corresponding stable global homotopy theory.
- Construct Hopkins-Kuhn-Ravenel character theory for $QEll^*_G(-)$;
- Set up quasi-theory for those with divisible group $\mathbb{G}_m\oplus(\mathbb{Q}/\mathbb{Z})^n$;
- Generalize the conclusions/constructions on $QEll^*_G(-)$ to elliptic cohomology theories;
  - Loop space construction;
  - Strickland's theorem;
- Explore the relation between $QEll^*_G(-)$ and physics;
- Explore the relation between elliptic cohomology theories and motivic homotopy theory.

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